

DUALITY OF TIME / CONSTRUCTIVE GAUGE THEORY

An Exact-Hamiltonian Completion-Band Framework for Auxiliary-Time Lattice Gauge Theory

Vacuum obstructions, protected spectral bundles,
quadratic-form Feshbach reduction, and a
constructive Yang–Mills research program

Fourth revised umbrella preprint with Kato-band geometry, phase compatibility, and
explicit no-extra-gauge criteria

Mohamed Haj Yousef

Physics Department, United Arab Emirates University, Al Ain, United Arab Emirates
mhajyousef@uaeu.ac.ae

Proof-status and publication-positioning certificate. This document is an *umbrella technical preprint*, not a claimed proof of the Yang–Mills Millennium problem. Its central completed results are: an exact finite-regulator Friedrichs Hamiltonian and vacuum Osterwalder–Schrader process; an exact covariant interruption functional and holonomy-twisted spectrum; a symmetry-protected no-go theorem for an unpunctured multiplication window; the corresponding punctured repair; a corrected operator Schur–Feshbach estimate; a quadratic-form Feshbach identity suitable for unbounded gauge Hamiltonians; and an explicit linear fast-sector scaling window.

The completion-band, renormalization-group, continuum, and universality stages are retained as a constructive research program. They are labeled *Program Requirements*, not theorems. In particular, the manuscript does not prove the nonlinear regulator-uniform fast-sector band, gauge-compatible global patching, a weak-to-massive four-dimensional RG crossover, a nontrivial continuum Osterwalder–Schrader limit, auxiliary invisibility, or intrinsic Yang–Mills universality.

The DTT interpretation motivates the auxiliary completion coordinate and the interruption-response language, but the technical theorems are stated so that they can be assessed independently of that interpretation. This edition also separates two mathematically different uses of spectral bands: a retained carrier band may support a Kato–Berry–Wilczek–Zee gauge connection, whereas a removable completion band must not leave an additional finite-energy gauge field. A separate focused manuscript extracted from this umbrella preprint contains the publishable no-go and form-Feshbach core without the broader chronogeometric program.

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Principal upgrades in this edition.

- (U1) The title and abstract now lead with the manuscript’s completed technical results rather than with an unexecuted renormalization program.
- (U2) Programmatic statements are labeled *Program Requirements*; the final section consolidates them by dependency and proof burden.
- (U3) The completion-band dynamics remains generated by an exact self-adjoint Schrödinger Hamiltonian and its Feynman–Kac semigroup.
- (U4) Orbit-type stratification and singular gauge-quotient geometry are made explicit in the slow–fast construction.
- (U5) The extensive holonomy-closure term is assigned a volume-uniform local polymer-norm decoupling requirement; pointwise coefficient decay is no longer treated as sufficient.
- (U6) Restoration of full four-dimensional Euclidean covariance and isotropic normalization is isolated as an independent continuum requirement.
- (U7) The global completion band must contain the vacuum exactly, and Berry, Born–Huang, and higher-rank band data remain explicit effective-action obligations.
- (U8) The technical theorem core is separated from the DTT interpretation, enabling a concise specialist submission.
- (U9) A protected-band theorem now records the correct Kato connection $\nabla^\Pi = \Pi d$, its curvature $\Pi(d\Pi \wedge d\Pi)\Pi$, and compatibility with a fixed positive-energy complex structure.
- (U10) Retained gauge-carrier bands are distinguished from auxiliary completion bands; the former may carry physical holonomy, while the latter must be rank one, flat, absorbed into the declared gauge data, or pushed to infinite energy.
- (U11) Optional inner-spectrum trace functionals are classified as matching diagnostics only; they may not replace a four-dimensional renormalization condition or introduce residual log-periodic running in pure Yang–Mills.

Abstract

We study an auxiliary-time lattice gauge model through exact finite-regulator Hamiltonian and spectral methods. The first main result is a vacuum obstruction: if a nonconstant multiplication window divides the strictly positive vacuum between its low and high parts, then the natural unpunctured relative-mixing inequality for the vacuum-corrected Schur–Feshbach map is impossible. We identify the exact offending terms and prove the corresponding punctured repair. The second main result is an exact continuous-time completion Hamiltonian, defined by a Friedrichs form and its Feynman–Kac semigroup, together with a reversible vacuum Osterwalder–Schrader process. The third main result is a quadratic-form Feshbach identity suitable for unbounded gauge Schrödinger operators and an explicit auxiliary scaling in which the linearized relative and holonomy stiffnesses diverge while the bare local closure coefficient vanishes.

The model contains a compact auxiliary cycle inside each ordinary Euclidean-time transfer slice. Spatial/auxiliary plaquettes define a nonnegative, gauge-invariant interruption functional that is exactly a Hilbert–Schmidt square of a covariant auxiliary difference. Auxiliary axial gauge yields a holonomy-twisted spectrum and rules out any mass-gap argument based solely on the first nonzero Fourier mode. To replace the failed hard-window strategy, we formulate local completion Hamiltonians with genuine spectral bands, a stratified slow–fast decomposition, gauge-compatible band patching, and form-level adiabatic elimination. A new protected-band layer gives every smooth isolated cluster its intrinsic Kato connection and makes a crucial role distinction: a retained carrier band may encode gauge holonomy, but a completion band introduced only to remove auxiliary modes must have trivial, absorbed, or infinitely costly geometric holonomy. The intrinsic connection is $\nabla^\Pi = \Pi d$; the often-written expression $\Pi d\Pi|_{\text{Ran } \Pi}$ vanishes because $\Pi(d\Pi)\Pi = 0$ and is not the connection.

The remaining steps are presented as program requirements: a nonlinear regulator-uniform completion band; a volume-uniform local polymer bound proving that the auxiliary closure device disappears from the four-dimensional effective action; scale-uniform lattice gauge rigidity and Poincaré inequalities; a gauge-covariant renormalization trajectory connecting asymptotic freedom to a massive infrared domain; restoration of full Euclidean covariance; construction of a nontrivial continuum Osterwalder–Schrader theory; auxiliary invisibility; and intrinsic pure-Yang–Mills universality. The DTT language of recurrent inner-time generation and outer-time observation supplies an interpretation of the auxiliary construction, but no interpretive premise is used in place of these estimates. The paper is therefore a rigorous framework and proof audit, not a completed Yang–Mills proof.

Claim-status convention. A statement labeled *Theorem*, *Proposition*, *Lemma*, or *Corollary* is proved in the manuscript from prior displayed hypotheses or from a specifically cited standard theorem. A statement labeled *Program Requirement* is a required new result that is not proved here. A *Proof Audit* records exactly why the preceding material does not establish a requirement. No program requirement is used as though it had already been proved.

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1 Introduction, completed results, and proof boundary

This article is organized around a completed technical theorem rather than around a claimed solution of four-dimensional Yang–Mills theory. The central obstruction concerns a natural low/high decomposition by a multiplication operator. A positive ground state is generally cut by such a window, and the rank-one vacuum correction in the gap operator changes the off-diagonal block. The resulting relative-mixing inequality fails strictly on the transported vacuum component.

Theorem 1.1 (Symmetry-protected hard-window obstruction). *Let T be a bounded positive self-adjoint operator, let Ω be a unit vector with $T\Omega = \Omega$, let $0 < r < 1$, and set*

$$G = rI - T + (1 - r)|\Omega\rangle\langle\Omega|.$$

Let P be an orthogonal projection, $Q = I - P$, and assume

$$\omega = P\Omega \neq 0, \quad \zeta = Q\Omega \neq 0, \quad D = QGQ \geq dQ \quad (d > 0).$$

Then

$$\langle \omega, PTQD^{-1}QTP\omega \rangle > \langle \omega, PGP\omega \rangle > 0. \quad (1)$$

Consequently no $\eta \leq 1$ can satisfy

$$PTQD^{-1}QTP \leq \eta PGP \quad (2)$$

on the whole low space $P\mathcal{H}$.

The proof is given in [section 12.2](#). The theorem has two consequences. First, the vacuum line must be transported exactly or removed before a strict relative estimate is formulated. Second, a local-to-global proof should not be based on classical zero sets of plaquette multiplication operators: those sets may have measure zero and therefore provide no nonzero local quantum ground vectors.

The second completed contribution is an exact finite-regulator Hamiltonian framework. The auxiliary-time model is generated by a Friedrichs Schrödinger operator, not by an assumed logarithmic identification with a single Strang step. Its positive semigroup has a unique ground state and a reversible two-sided vacuum process. The third contribution is the form-level Schur–Feshbach identity of [theorem 18.3](#), which is adapted to unbounded gauge Hamiltonians. A frozen auxiliary scaling also yields a genuine *linearized* fast-sector window: relative auxiliary modes and the holonomy mean mode acquire diverging tangent stiffness, while the bare local closure coefficient tends to zero.

1.1 Main results and novelty boundary

Result	Status	Mathematical role
Vacuum obstruction for an unpunctured hard window	Proved	Rules out a natural but false relative-mixing estimate and identifies the transported-vacuum correction exactly.
Punctured vacuum-compatible Schur–Feshbach criterion	Proved	Gives the correct finite-regulator reduction after removal of the transported vacuum line.
Exact Friedrichs Hamiltonian and vacuum OS process	Proved	Supplies a common self-adjoint generator, positive semigroup, unique vacuum, and reflection-positive vacuum representation.
Quadratic-form Feshbach identity	Proved	Replaces an inappropriate bounded-block assumption for unbounded gauge Schrödinger operators.
Linear auxiliary fast-scale window	Proved	Shows divergence of tangent relative and holonomy stiffness for one frozen scaling; it is not a nonlinear spectral-band theorem.
Nonlinear band, four-dimensional RG, continuum, and universality	Open program	These contain the model-specific and Yang–Mills-specific estimates required for a full solution.

The standard ingredients are compact-group heat kernels, Friedrichs extensions, Perron–Frobenius theory, the Osterwalder–Schrader construction, Riesz projections, and Schur–Feshbach identities. The model-specific contributions are the covariant interruption observable, the exact hard-window obstruction, the punctured repair, and the proposed completion-band regulator with its explicit scaling analysis.

1.2 Technical independence from the DTT interpretation

The DTT language motivates the distinction between an auxiliary recurrent completion coordinate and the outer transfer time. It also motivates the phrase *inner-flow interruption* for nonzero mixed covariant curvature. None of the finite-regulator, spectral, or form-theoretic theorems depends on accepting an ontological interpretation of inner time. The quaternionic readout and Lorentzian record-cone sections are therefore structural background, not premises in the gauge-theory estimates.

1.3 Scope of this umbrella preprint

The paper keeps the complete dependency graph because its purpose is to identify exactly what a constructive completion would require. Program requirements appear at the points where they are consumed and are consolidated in [section 32](#). A separate focused technical manuscript extracts the no-go theorem, exact Hamiltonian, punctured reduction, and form-Feshbach results for specialist review.

1.4 The exact endpoint

Let G be a compact connected Lie group with simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{W}_0$ be the countable gauge-invariant Euclidean core defined in [section 27](#). The endpoint of the construction is the following statement.

Program Requirement 1.2 (Four-dimensional Yang–Mills existence and mass gap). For every such G , construct a continuous Euclidean functional

$$\mathcal{S}_G : \mathfrak{W} \longrightarrow \mathbb{C}$$

on the completion of $\mathfrak{W}_0$, satisfying normalization, Euclidean covariance, permutation symmetry, reflection positivity, regularity, locality after reconstruction, clustering, and nontriviality. Its Osterwalder–Schrader reconstruction must give a Hilbert space $\mathcal{H}_G$, a unique vacuum Ω_G , a local gauge-invariant observable net, and a nonnegative Hamiltonian H_G such that

$$H_G \Omega_G = 0, \quad \text{Spec}\left(H_G|_{\Omega_G^\perp}\right) \subseteq [\Delta_G, \infty) \quad (3)$$

for some $\Delta_G > 0$. The construction must be intrinsically associated with pure Yang–Mills and independent of the admissible auxiliary regulator.

Equation (3) is a program requirement, not a result of the present manuscript. The purpose of the paper is to define a single route to it, prove every formal implication in that route, and isolate the genuinely new analytical statements that remain.

1.5 What is proved here

The following components are established in full.

- (P1) Under explicit finite-dimensional algebraic and record-covariance hypotheses, the local displacement carrier is quaternionic and its center-insensitive readout is $W \simeq \mathbb{R}^3$.
- (P2) Under explicit completion-order and soldering hypotheses, the observer-level record cone is the rank-two spin-factor cone and determines a Lorentzian conformal structure.
- (P3) Compact inner-time phase symmetry and Lorentzian causal symmetry are separated representation-theoretically.
- (P4) A frozen auxiliary-time lattice gauge model has an injective, compact, positivity-improving heat-kernel transfer operator, a unique positive vacuum, a well-defined Hamiltonian, and a two-sided vacuum OS process. The upgraded completion-band regulator is separately defined by an exact Friedrichs Schrödinger Hamiltonian and its exact semigroup.
- (P5) Mixed plaquettes are exact Hilbert–Schmidt squares of covariant inner-time differences. Their holonomy-twisted spectrum and zero set are determined exactly.
- (P6) The global hard-window vacuum obstruction and the corrected punctured Feshbach theorem are proved.
- (P7) A normalized linear completion model has a transverse stiffness bounded below by the minimum of its nonzero Fourier stiffness and its holonomy-closure stiffness. For the frozen scaling of section 7.4, both tangent stiffnesses diverge in physical units.
- (P8) Abstract spectral-band, operator and quadratic-form Feshbach, IMS-localization, Poincaré-to-gap, exponential-decay-to-gap, and Mosco gap-transfer theorems are proved; the operator resolvent estimate is stated with all Schur-complement terms.
- (P9) Equality of continuous OS functionals on the explicit countable core implies unitary equivalence of the reconstructed vacuum theories.
- (P10) A smooth isolated spectral cluster defines a Hermitian vector bundle with Kato connection and curvature expressed directly through its Riesz projection. For a fixed symmetry-compatible positive-energy complex structure, the band projection and all invariant Feshbach reductions are complex-linear. This standard bundle theorem is used to separate retained gauge-carrier bands from removable completion bands.

1.6 What remains to be proved

The proof is not complete until four model-specific packages have been established.

- (M1) **Exact slow–fast reduction and nonlinear completion band.** Disintegrate the exact completion Hamiltonian into physical base and auxiliary fast forms, prove a regulator-uniform fast spectral band in every holonomy, boundary, and flux stratum, classify its Kato holonomy, and control Berry/Born–Huang terms without confusing a retained gauge-carrier band with a removable completion band.
- (M2) **Four-dimensional nonabelian coercivity.** After eliminating the completion band, prove a block Poincaré inequality and a gauge-covariant RG trajectory connecting asymptotic freedom to a massive infrared domain.
- (M3) **Constructive continuum limit.** Construct nontrivial limiting Schwinger functions and prove the full OS package and gap-preserving comparison-space convergence.
- (M4) **Auxiliary invisibility and universality.** Prove divergence of every purely auxiliary excitation, scalarization of neutral auxiliary observables, disappearance of Berry data, and intrinsic pure-Yang–Mills universality.

These are not missing editorial details. Package (M2), in particular, contains the genuinely nonabelian infrared theorem.

1.7 Main conditional implication

The whole paper is organized to prove the following exact implication.

Theorem 1.3 (Completion-band proof-transfer theorem). *Fix G and the regulator family of section 7. Assume that the following theorem packages are proved for that family:*

- (H1) *the slow–fast disintegration, nonlinear local band, protected-band role, Berry-control, and global band results in theorems 5.7, 14.6, 15.4, 16.4, 16.6, 16.8 and 17.3;*
- (H2) *the scaling-window, spectral-correction irrelevance, effective-action, and gauge-covariant RG results in theorems 19.4, 19.5, 22.1, 23.1, 23.3 and 24.2;*
- (H3) *the massive-domain clustering and multiscale Poincaré results in theorems 25.1 and 26.1;*
- (H4) *the continuum OS and comparison-space convergence results in theorems 28.4 and 29.3;*
- (H5) *the auxiliary-invisibility and intrinsic-identification results in theorems 30.2 and 31.1.*

Then the functional constructed in theorem 28.4 satisfies theorem 1.2 with a positive gap Δ_G supplied by theorem 26.1.

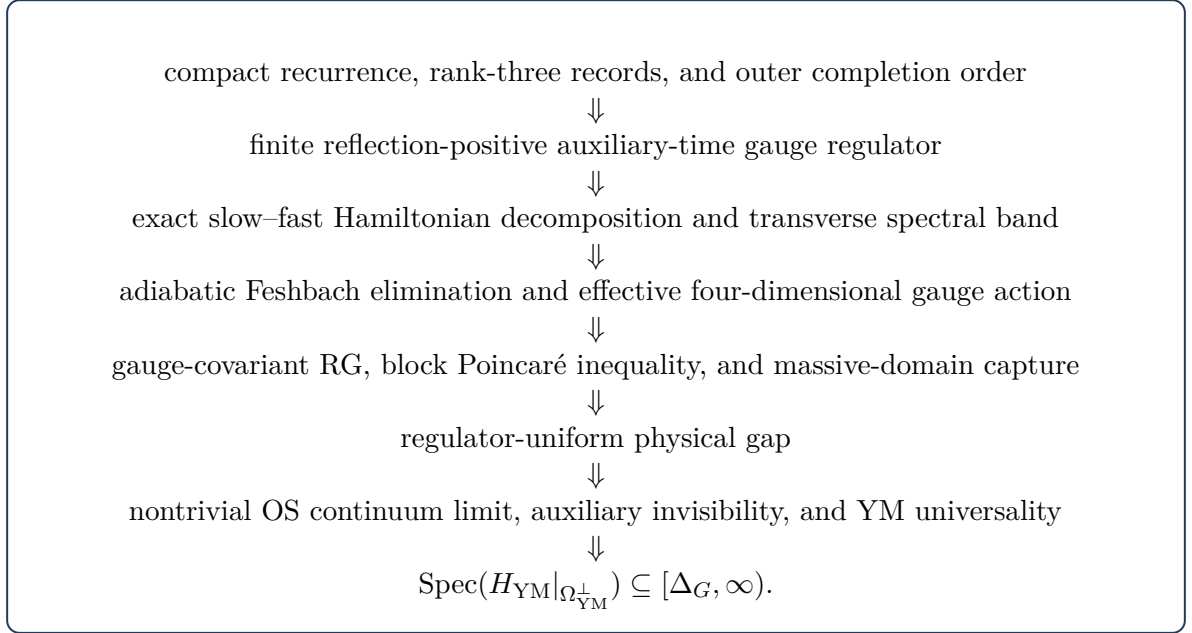
Proof. The finite-regulator transfer and OS theory are constructed in sections 8 and 9. The completion-band hypotheses and the abstract adiabatic theorem of section 18 identify the low-energy DTT dynamics with a quasi-local four-dimensional effective gauge Hamiltonian. The RG, clustering, and Poincaré hypotheses give a regulator-uniform lower bound

$$H_n^{\text{cb}} \geq \Delta_G \quad \text{on } (\Omega_n^{\text{cb}})^\perp.$$

The comparison-space theorem in section 29 transfers the same lower bound to the continuum Hamiltonian. The auxiliary-invisibility theorem identifies the continuum physical net with its four-dimensional neutral subtheory, and the intrinsic-identification theorem verifies the Yang–Mills field content, Ward identities, ultraviolet normalization, and universality. The continuum OS reconstruction therefore yields the pure Yang–Mills Hamiltonian, and the transferred lower bound is exactly (3). $\square$

The theorem is a complete logical implication. Its hypotheses are not asserted to have been proved.

1.8 Dependency map



The specifically DTT step is the construction and elimination of a transverse completion band. The specifically Yang–Mills step is the nonabelian multiscale coercivity of the effective four-dimensional theory. Neither may be substituted for the other.

1.9 Organization

Sections 2 to 6 establish the symmetry, protected-band, and record-geometric architecture. Sections 7 to 10 construct and analyze the finite regulator. Sections 11 and 12 give the interruption response, the vacuum-compatible Feshbach map, and the hard-window no-go theorem. Sections 14 to 19 develop the completion-band redesign. Sections 20 to 26 specify the genuinely nonabelian proof engine. Sections 27 to 31 treat the continuum and identification stages. The final section gives the exact status ledger and the shortest proof-production order.

Part I

Structural background and exact finite-regulator theory

Logical separation. The finite-regulator gauge construction begins in section 7. Its definitions and proofs do not require the quaternionic or chronogeometric interpretation in sections 3 and 6. Those sections explain the broader DTT motivation and the separation of phase and causal roles; they are not used as substitutes for gauge-theory coercivity.

2 Formal architecture, symmetry levels, and source discipline

2.1 Outer time and auxiliary time

Outer time is the transfer direction. The positive operator T_n propagates one outer-time step $a_{t,n}$, and reflection positivity is imposed across an outer-time hyperplane. The auxiliary coordinate $\tau \in C_{N_{\tau,n}}$ lies inside a single transfer slice. Its cyclic translation is an internal symmetry, not the physical Hamiltonian evolution. These roles are never interchanged.

2.2 Four non-equivalent symmetry layers

The attached symmetry manuscripts use the word symmetry for several structures that must not be collapsed into one another. The present proof audit distinguishes the following levels.

Definition 2.1 (Symmetry hierarchy). At finite regulator n , the relevant symmetry data are:

- (S1) the compact inner-time recurrence group $C_{N_{\tau,n}}$, represented by the unitary shift $R_{\tau,n}$;
- (S2) the compact slice gauge group $\mathcal{G}_n$, whose invariant subspace defines the gauge-reduced Hilbert space rather than an ordinary global-symmetry multiplet;
- (S3) a compact regulator symmetry group $\mathcal{S}_n$, generated by those spatial translations, cubic rotations, center/flux transformations, and inner shifts that preserve the frozen boundary sector and commute with T_n and K_n ;
- (S4) the outer-time reflection and positive transfer semigroup, which reconstruct the physical Hamiltonian and, only after a Lorentzian continuum reconstruction, the proper orthochronous causal symmetry.

The same operator must not be interpreted simultaneously as a generator of all four levels. In particular, $R_{\tau,n}$ is periodic and unitary, whereas T_n is a positive contraction. Gauge invariance is a constraint on physical states, compact recurrence organizes phase sectors, and Lorentz symmetry belongs to the observer-level causal readout.

2.3 Observation as a symmetry filter

The record-oriented drafts describe observation as a transition from a richer latent completion structure to a stable public order; the published dual-time topology paper supplies complementary connection, projection, and hidden-channel language [6, 50]. The finite-regulator analogue is a group average rather than an assumed measurement postulate. For bounded operators on the slice Hilbert space define

$$\mathbb{E}_{\tau,n}(A) = \frac{1}{N_{\tau,n}} \sum_{j=0}^{N_{\tau,n}-1} R_{\tau,n}^j A R_{\tau,n}^{-j}. \quad (4)$$

Proposition 2.2 (Inner-time observation filter). *The map $\mathbb{E}_{\tau,n}$ is a normal unital completely positive idempotent map onto the commutant of the cyclic inner-time action. On vectors, the corresponding average*

$$P_{0,n} = \frac{1}{N_{\tau,n}} \sum_{j=0}^{N_{\tau,n}-1} R_{\tau,n}^j$$

is the orthogonal projection onto the inner-translation-invariant subspace. If an operator commutes with $R_{\tau,n}$, then it is fixed by $\mathbb{E}_{\tau,n}$ and reduces $P_{0,n} \mathcal{H}_n^{\text{sl}}$.

Proof. Equation (4) is the finite-group Reynolds average. Each summand is a unital $*$ -automorphism, so the average is unital and completely positive. Reindexing the finite sum

proves idempotence and shows that its range is precisely the fixed-point algebra. The vector statement follows because a finite average of a unitary representation is the orthogonal projection onto its invariant vectors. The final assertion is immediate. $\square$

This is a rigorous version of *symmetry filtering*. It does not by itself describe spontaneous symmetry breaking: the averaged dynamics still commutes with the inner-time symmetry. A state that fails to be invariant in an infinite-volume limit, or a parent theory with degenerate temporal vacua, would require an additional order-parameter and phase-selection theorem.

Warning 2.3 (No primitive $t \leftrightarrow \tau$ exchange symmetry). Some attached dual-time drafts introduce a pre-observation interchange symmetry between inner and outer temporal roles and a gauge-invariant phase-gradient order parameter. The frozen Yang–Mills regulator used here does not possess such an exact coordinate exchange: outer time is the reflection/transfer direction and inner time is a compact completion coordinate. Accordingly, no quartic temporal symmetry-breaking potential is imported into the proof. The mathematically controlled remnants are role selection, future orientation, the conditional expectation (4), and the covariant interruption defect defined in section 11.

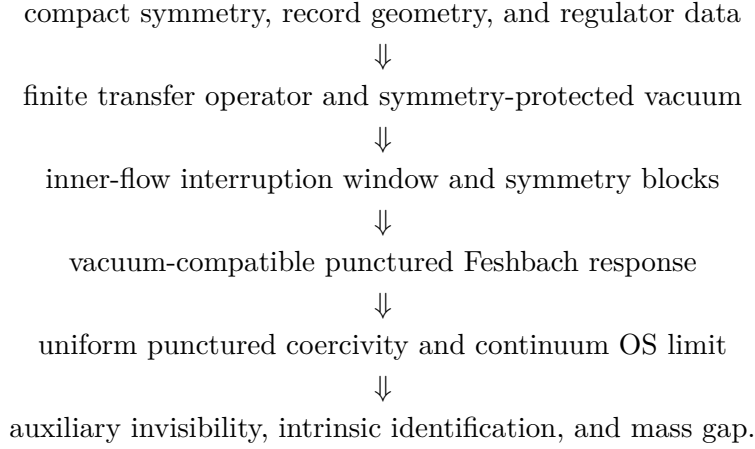
2.4 Permitted logical moves

The audit uses the following rules.

- (i) A fixed-regulator spectral gap obtained from compactness may depend arbitrarily on n ; it is not a physical mass gap.
- (ii) A two-loop formula may define a bare numerical sequence, but it does not prove convergence of a nonperturbative measure.
- (iii) The spectrum of a configuration-dependent auxiliary difference operator is not the spectrum of K_n , T_n , or H_n .
- (iv) A classical zero set is not automatically a quantum local ground space in L^2 . In the present continuous-group multiplication model, the exact zero set has Haar measure zero.
- (v) The trivial-holonomy zero-mixed-curvature stratum may be identified with ordinary lattice Yang–Mills configurations, but that identification supplies no infrared estimate.
- (vi) A Schur complement identifies the exact effective operator; it does not determine its sign.
- (vii) Compact inner-time translation may remain an exact symmetry while individual configurations carry a nonzero covariant interruption defect. Such a defect is not spontaneous breaking of the Yang–Mills gauge symmetry.
- (viii) A pre-observation role-exchange symmetry or quartic temporal order-parameter potential is not part of the frozen regulator unless separately defined and proved compatible with reflection positivity.
- (ix) No continuum object is named YM until its correlation functions, observable net, and dynamics have been identified without using that name as a definition.

2.5 Dependency order

The only admissible order is



The hard-window branch reaches the Feshbach map and then disproves the unpunctured mixing premise. The completion-band branch developed in Part II is a distinct redesign: it replaces the multiplication window by genuine spectral bands and proves only the abstract consequences of the still-targeted uniform band and RG estimates. No unproved arrow in either branch is treated as established.

3 Quaternionic rank-three record geometry

The role of this section is deliberately limited. It provides one mathematically explicit route from recurrent phase composition to an oriented three-dimensional local displacement carrier. It does not prove a field equation, a continuum limit, or a mass gap. The construction is included because it explains why the spatial lattice used later may be regarded as an observer-level record space rather than as an unexplained primitive.

3.1 Algebraic and record hypotheses

Assumption 3.1 (Quaternionic record package). Let A be a finite-dimensional unital associative real algebra. Assume:

(Q1) there is a positive-definite quadratic norm $N : A \rightarrow \mathbb{R}_{\geq 0}$ such that

$$N(ab) = N(a)N(b), \quad N(1) = 1;$$

(Q2) A contains an element j with $j^2 = -1$, representing a selected recurrent phase;

(Q3) A contains another element k with $k^2 = -1$ and $jk \neq kj$;

(Q4) the connected inner automorphism group $\text{Inn}_0(A)$ acts on a nonzero real record space $\mathcal{R} \subseteq A^*$, the record space annihilates the center, and it is invariant under the dual inner action;

(Q5) the record family is operationally complete: its common kernel is exactly $Z(A)$.

The last condition means that two algebra elements have the same observable displacement record precisely when they differ by a central scalar component.

Lemma 3.2 (Division property). *Under theorem 3.1, every nonzero element of A is invertible.*

Proof. If $a \neq 0$, then $N(a) > 0$. If $ab = 0$, multiplicativity gives

$$0 = N(ab) = N(a)N(b),$$

so $N(b) = 0$ and hence $b = 0$. Thus left multiplication by a is injective. Since A is finite dimensional, it is surjective, and there is b with $ab = 1$. The same argument for right multiplication gives c with $ca = 1$; associativity implies $b = c$. Therefore a is invertible. $\square$

Theorem 3.3 (Quaternionic carrier and rank-three readout). *Under theorem 3.1, the normed algebra A is isomorphic to the quaternion algebra $\mathbb{H}$. The intrinsic record map*

$$r_{\mathcal{R}} : A \longrightarrow \mathcal{R}^*, \quad r_{\mathcal{R}}(a)(\ell) = \ell(a),$$

has kernel $Z(A)$, and its image is naturally isomorphic, as a real representation of $\text{Inn}_0(A)$, to

$$W := A/Z(A) \simeq \mathbb{H}/\mathbb{R}1 \simeq \text{Im } \mathbb{H} \simeq \mathbb{R}^3.$$

Proof. By theorem 3.2, A is a finite-dimensional associative real division algebra. Frobenius' classification gives

$$A \simeq \mathbb{R}, \quad \mathbb{C}, \quad \text{or} \quad \mathbb{H}.$$

The existence of $j^2 = -1$ excludes $\mathbb{R}$, while the noncommuting pair j, k excludes $\mathbb{C}$. Hence $A \simeq \mathbb{H}$. The common kernel of the record family is $Z(A)$ by operational completeness, so the first isomorphism theorem gives $\text{im } r_{\mathcal{R}} \simeq A/Z(A)$. For $A \simeq \mathbb{H}$, the center is $\mathbb{R}1$, and every coset has a unique purely imaginary representative. This identifies the quotient linearly and equivariantly with $\text{Im } \mathbb{H}$, which has real basis i, j, k . $\square$

3.2 Metric, orientation, and the direction sphere

The quotient $W = A/Z(A)$ is a real vector space and representation, not a quotient algebra. Let

$$s : W \longrightarrow \text{Im } \mathbb{H}$$

be the canonical map selecting the unique purely imaginary representative.

Proposition 3.4 (Quaternionic metric and cross product). *There are unique bilinear maps*

$$\langle \cdot, \cdot \rangle : W \times W \rightarrow \mathbb{R}, \quad \times : W \times W \rightarrow W$$

such that

$$s(u)s(v) = -\langle u, v \rangle + s(u \times v). \quad (5)$$

The first is a positive-definite Euclidean inner product and the second is the associated oriented cross product. In particular,

$$\|u \times v\|^2 = \|u\|^2 \|v\|^2 - \langle u, v \rangle^2.$$

Proof. For purely imaginary quaternions x, y , the real part of xy is symmetric and the imaginary part is antisymmetric. Set

$$\langle u, v \rangle = -\text{Re}(s(u)s(v)), \quad s(u \times v) = \text{Im}(s(u)s(v)).$$

The quaternion norm gives $N(x) = x\bar{x} = \|x\|^2$, so the first form is the usual Euclidean inner product on $\text{Im } \mathbb{H}$. Expanding $N(xy) = N(x)N(y)$ and separating scalar and imaginary parts gives the displayed cross-product identity. The ordered basis (i, j, k) fixes the orientation through $ij = k$. $\square$

Proposition 3.5 (Rotations and Hopf directions). *Conjugation by unit quaternions induces the standard double covering*

$$\text{Sp}(1) \longrightarrow SO(W), \quad q : u \longmapsto s^{-1}(q s(u) q^{-1}),$$

with kernel $\{\pm 1\}$. Fixing a unit phase $j \in \text{Im } \mathbb{H}$, the unit sphere of directions is naturally

$$S(W) \simeq S^3/U(1)_j \simeq \mathbb{C}P^1 \simeq S^2.$$

Proof. Quaternionic conjugation preserves real part and norm, so it acts orthogonally on $\text{Im } \mathbb{H}$. The action preserves orientation and has kernel $\{\pm 1\}$; the usual axis–angle parametrization gives surjectivity onto $SO(3)$. The right action of $U(1)_j = \{\exp(j\theta)\}$ on unit quaternions is free, and the map $q \mapsto -jqj^{-1}$ is constant on right-phase orbits and surjective onto the unit imaginary quaternions. This is the Hopf fibration. $\square$

Proof Audit 3.6 (Scope of the rank-three theorem). The theorem selects a local oriented Euclidean carrier under strong constitutive hypotheses. It does not prove that these hypotheses are uniquely forced by a microscopic theory, does not construct global spacetime, and supplies no lower bound for the Yang–Mills Hamiltonian. The later gauge construction can be formulated on an ordinary cubic lattice even if this section is omitted. Its role is explanatory and organizational.

4 Compact symmetry, Real commutants, and causal separation

The compact-symmetry manuscripts supply two ingredients that are directly useful for the low-window problem: a block calculus for symmetry-compatible operators and a theorem preventing compact phase geometry from being confused with Lorentzian causal geometry [3–5].

4.1 Real isotypic blocks and carrier algebras

Let $U_n : \mathcal{S}_n \rightarrow O(\mathcal{K}_{n,\mathbb{R}})$ be a strongly continuous orthogonal representation of a compact regulator symmetry group on a real Hilbert space. For a finite-dimensional $\mathcal{S}_n$ -invariant subspace $E \subset \mathcal{K}_{n,\mathbb{R}}$, complete reducibility and the real Schur lemma give the following standard form.

Theorem 4.1 (Real compact-symmetry block form). *There is an orthogonal decomposition*

$$E \cong \bigoplus_{\sigma \in \Sigma_E} V_\sigma \otimes_{\mathbb{D}_\sigma} M_\sigma, \quad \mathbb{D}_\sigma := \text{End}_{\mathcal{S}_n}(V_\sigma) \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}, \quad (6)$$

where V_σ is an irreducible real $\mathcal{S}_n$ -module and M_σ is its multiplicity space. Every self-adjoint operator A on E commuting with $\mathcal{S}_n$ has the form

$$A = \bigoplus_{\sigma \in \Sigma_E} I_{V_\sigma} \otimes A_\sigma, \quad A_\sigma \in \text{End}_{\mathbb{D}_\sigma}(M_\sigma), \quad A_\sigma = A_\sigma^*. \quad (7)$$

Consequently, $A \geq cI$ if and only if $A_\sigma \geq cI$ on every multiplicity block.

Proof. Average an inner product over $\mathcal{S}_n$ and decompose E into real isotypic components. On an irreducible real component, the commutant is a finite-dimensional real division algebra and is therefore $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The standard multiplicity-space description identifies the equivariant endomorphisms of an isotypic component with $\mathbb{D}_\sigma$ -linear endomorphisms of M_σ . Self-adjointness and order are tested blockwise. $\square$

Corollary 4.2 (Symmetry reduction of a dangerous cluster). *Suppose T_n , K_n , and $P_{\Omega,n}$ commute with $\mathcal{S}_n$, and let Π_n be a finite-rank $\mathcal{S}_n$ -invariant dangerous-cluster projection. Then every Feshbach operator built from these data commutes with $\mathcal{S}_n$, and its restriction to $\Pi_n \mathcal{H}$ reduces to the multiplicity matrices in (7). Real-, complex-, and quaternionic-type sectors can therefore be tested independently.*

Proof. Functional calculus preserves commutation, as do sums, products, compressions by invariant projections, and inverses on invariant reducing subspaces. Apply theorem 4.1. $\square$

On a complex-type irreducible, invariant orthogonal complex structures form the sign pair $\{\pm J\}$; on a real-type irreducible there is none; on a quaternionic-type irreducible there is an S^2 -family. Periodic S^1 symmetry removes the sign ambiguity after a positive-frequency orientation is selected, while a finite cyclic approximation requires a compatible orientation of paired characters. This is the representation-theoretic sense in which inner-time recurrence can generate complex kinematics without yet generating a mass scale.

4.2 Compact phase symmetry cannot itself be Lorentzian

Theorem 4.3 (Separation of compact phase and causal forms). *Let $V \neq 0$ be an irreducible real orthogonal representation of a compact group S . Every continuous S -invariant symmetric bilinear form on V is a real scalar multiple of the invariant positive-definite inner product. Hence no nondegenerate S -invariant split-signature form exists on V . A complex-type sector may carry an invariant orthogonal complex structure, but it cannot carry an invariant Lorentzian metric on the same irreducible carrier.*

Proof. Represent an invariant symmetric bilinear form as $B(x, y) = \langle x, Ay \rangle$ relative to the invariant inner product. Then $A = A^*$ and A commutes with S . In finite dimension, A has a real eigenvalue λ . The intertwiner $A - \lambda I$ is noninvertible. By irreducibility and the real Schur lemma, every nonzero intertwiner is invertible, so $A = \lambda I$. The signature is therefore definite up to an overall sign. $\square$

Remark 4.4 (Where Lorentzian symmetry may occur). The theorem does not forbid a split form on a reducible representation, a multiplicity space, or a reconstructed record bundle. It says that compact phase symmetry and Lorentzian causal geometry must occur at different structural levels. This distinction is essential below: inner-time interruption is measured by a positive compact-sector defect, whereas outer-time causality is carried by the soldered completion cone.

5 Protected spectral bundles, Kato holonomy, and band-role separation

The protected-eigenspace construction developed in the companion inner-time spectral-geometry manuscript is useful here only after a precise distinction is made. An isolated fast spectral cluster always carries an adiabatic vector-bundle connection; whether that connection is a desired physical gauge field or an unwanted remnant depends on the role of the band. The general statement is standard, but its use in the completion-band program is structurally decisive [38–41].

Definition 5.1 (Smooth isolated spectral band). Let M be a smooth parameter manifold and let $u \mapsto K(u)$ be a C^1 family of self-adjoint operators on a fixed complex Hilbert space with common form domain. Assume that a positively oriented contour Γ lies in the resolvent set of every $K(u)$ and encloses a finite spectral cluster of constant rank m . Define

$$\Pi(u) = \frac{1}{2\pi i} \int_{\Gamma} (z - K(u))^{-1} dz, \quad E_u = \text{Ran } \Pi(u). \quad (8)$$

The family $E = \bigsqcup_{u \in M} E_u \rightarrow M$ is called the protected band bundle.

Theorem 5.2 (Kato connection of a protected band). *Under theorem 5.1, $E \rightarrow M$ is a rank- m Hermitian vector bundle. Its intrinsic adiabatic connection is*

$$\nabla^{\Pi} s = \Pi ds \quad (9)$$

for smooth sections s of E . In a local orthonormal frame $e_1, \dots, e_m$, the skew-Hermitian connection matrix and curvature are

$$\mathcal{A}_{ab} = \langle e_a, de_b \rangle, \quad \mathcal{F}^\Pi = \Pi(d\Pi \wedge d\Pi)\Pi. \quad (10)$$

Parallel transport is unitary, so the holonomy group is a subgroup of $U(m)$.

Proof. Norm differentiability of the resolvent integral gives a C^1 projection field of constant finite rank, hence a Hermitian subbundle. For sections s, t with $\Pi s = s$ and $\Pi t = t$,

$$d\langle s, t \rangle = \langle \Pi ds, t \rangle + \langle s, \Pi dt \rangle,$$

so ∇^Π is metric compatible. In a local orthonormal frame its matrix is $\mathcal{A}_{ab} = \langle e_a, de_b \rangle$, which is skew-Hermitian by differentiation of $\langle e_a, e_b \rangle = \delta_{ab}$. The standard projection calculation gives $(\nabla^\Pi)^2 = \Pi(d\Pi \wedge d\Pi)\Pi$. Metric compatibility makes the parallel transport unitary. $\square$

Warning 5.3 (The connection is not $\Pi d\Pi|_E$). Differentiating $\Pi^2 = \Pi$ gives $\Pi(d\Pi)\Pi = 0$. Hence $\Pi d\Pi|_E$ is identically zero and cannot be the adiabatic connection. The correct coordinate-free connection is $\nabla^\Pi = \Pi d$ acting on sections; its curvature is the projection formula in (10).

Proposition 5.4 (Compatibility with a fixed positive-energy complex structure). *Let the underlying real Hilbert bundle be trivial and let J be a fixed bounded orthogonal complex structure. Suppose $J \text{Dom } K(u) \subseteq \text{Dom } K(u)$ and $K(u)J = JK(u)$ for every u . Then*

$$[\Pi(u), J] = 0, \quad \nabla^\Pi(Js) = J\nabla^\Pi s.$$

If, in addition, the slow and fast forms and their reducing projections commute with J , then the associated operator or quadratic-form Feshbach reductions are J -linear.

Proof. The resolvent of $K(u)$ commutes with J , so the Riesz integral (8) does as well. Since J is independent of u ,

$$\nabla^\Pi(Js) = \Pi d(Js) = \Pi J ds = J\Pi ds.$$

The Feshbach assertion follows because sums, compressions, inverses on reducing subspaces, and Riesz representatives of J -invariant forms preserve commutation with J . $\square$

Definition 5.5 (Carrier band and completion band). A *carrier band* is retained in the physical low-energy theory; its Kato holonomy may represent physical internal gauge structure. A *completion band* is introduced only to isolate and eliminate auxiliary inner-time excitations; its geometric connection must therefore become trivial, scalar, absorbed into the already declared gauge data, or energetically inaccessible in the continuum limit.

Proposition 5.6 (Conditional rank reductions for retained carrier bands). *Let $E \rightarrow M$ be a retained protected band bundle with Kato connection.*

- (i) *If $\text{rank}_{\mathbb{C}} E = 1$, its structure group is $U(1)$.*
- (ii) *If $\text{rank}_{\mathbb{C}} E = 2$ and E carries a parallel antiunitary map J with $J^2 = -I$, the structure group reduces to $Sp(1) \simeq SU(2)$.*
- (iii) *If $\text{rank}_{\mathbb{C}} E = 3$ and the connection preserves a nowhere-vanishing complex volume form, the structure group reduces to $SU(3)$.*

Proof. The first statement is the unitary automorphism group of a Hermitian line. In complex rank two, unitary automorphisms commuting with a quaternionic structure form $Sp(1)$. In complex rank three, preserving a complex volume form imposes determinant one on the unitary holonomy, giving $SU(3)$. $\square$

Program Requirement 5.7 (Band-role and no-extra-gauge compatibility). For the pure Yang–Mills completion regulator, prove one of the following for every low-energy completion band: it is rank one with a globally positive real generator; its Kato curvature tends to zero in a volume-uniform local norm; its holonomy is a scalar flat connection absorbed into declared topological data; or the entire nontrivial holonomy sector has energy tending to infinity. If a retained carrier-band branch is used to generate the prescribed gauge group G , prove that its holonomy reduces exactly to G and coincides with the connection entering the Wilson-loop/curvature core. No additional $U(m)$ factor may survive merely because an auxiliary band is degenerate.

Remark 5.8 (Two distinct triplets). The real spatial triplet $W \simeq \text{Im } \mathbb{H} \simeq \mathbb{R}^3$ and a complex rank-three carrier band are different objects. The former is an observer-level displacement module; the latter, if present, is an internal Hermitian bundle whose volume-preserving holonomy may reduce to $SU(3)$. Equality of the number three does not identify the two sectors.

Proof Audit 5.9 (What protected-band gauge geometry does not prove). The existence of a Kato connection is automatic once a smooth isolated cluster has been supplied. The rank-(1, 2, 3) reductions are conditional on the existence of those ranks and the stated parallel structures. They do not derive the cluster pattern, Yang–Mills dynamics, asymptotic freedom, or a mass gap. For the Clay-facing fixed-group problem, the gauge group is prescribed in advance; the immediate role of [theorem 5.2](#) is to detect and eliminate unintended band holonomy, not to replace the nonabelian coercivity theorem.

6 Symmetric completion order and Lorentzian outer-record geometry

The chronogeometry draft reconstructs Lorentzian symmetry from stable completed records rather than identifying it with the compact inner-time phase geometry [1, 6]. The minimum kinematic bridge needed here is recorded as an explicit conditional theorem.

Assumption 6.1 (Record-geometric bridge). Let M be a smooth connected manifold. Assume:

- (C1) an oriented real clock line bundle $L \rightarrow M$ with an exact positive additive duration coordinate;
- (C2) an oriented Euclidean rank-three record bundle $W \rightarrow M$, locally identified with the quaternionic center-blind readout $\text{Im } \mathbb{H}$;
- (C3) a smooth soldering isomorphism $\sigma : L \oplus W \rightarrow TM$;
- (C4) a proper homogeneous self-dual completion cone $C_{\text{rec}}^+ \subset L \oplus W$, self-dual for a specified positive pairing, together with an order unit e and an $SO(W)$ -action that preserves the cone and pairing, fixes e , and acts irreducibly on W .

Theorem 6.2 (Spin-factor reconstruction of outer causal symmetry). *Under [theorem 6.1](#), $\dim M = 4$. After one positive rescaling of the length unit on W , the Euclidean Jordan product on each record fiber is*

$$(t, x) \circ (s, y) = (ts + \langle x, y \rangle, ty + sx), \quad (11)$$

and the cone of squares is

$$C_{\text{sf}}^+ = \{(T, x) \in \mathbb{R} \oplus W : T \geq 0, \|x\| \leq T\}. \quad (12)$$

Its determinant

$$q(T, x) = T^2 - \|x\|^2 \quad (13)$$

has Lorentzian signature, and $\sigma(C_{\text{sf}}^+) \subset TM$ determines a time-oriented Lorentzian conformal class. After a clock and length calibration, the cone automorphisms reduce to the proper orthochronous Lorentz group.

Proof. The soldering isomorphism gives $\dim M = \text{rank}(L \oplus W) = 1 + 3$. By the Koecher–Vinberg correspondence, a finite-dimensional homogeneous self-dual order-unit cone is the cone of squares of a Euclidean Jordan algebra. The stabilizer $SO(W)$ acts by Jordan automorphisms. Since

$$\text{Sym}^2(W) \cong \mathbb{R} \oplus S_0^2(W)$$

contains one trivial summand and no copy of W , equivariance forces $x \circ y = \alpha \langle x, y \rangle e$ with $\alpha > 0$; rescale W to obtain (11). The spectral idempotents $\frac{1}{2}(e \pm x/\|x\|)$ give (12). The standard Hermitian realization in $\text{Herm}_2(\mathbb{C})$ has determinant (13); cone-preserving transition maps are positive Lorentz similitudes, so the soldered cone determines a conformal Lorentzian metric. Calibration fixes the remaining scale [9]. $\square$

Proof Audit 6.3 (Kinematics is not a mass-gap estimate). Theorem 6.2 reconstructs the observer-level causal arena and its symmetry. It does not determine a matter action, a glueball spectrum, or a lower bound for the Yang–Mills Hamiltonian. The connection with mass enters only through the interruption-response mechanism of section 11.

7 Frozen group data, regulator family, sectors, and scale

7.1 Compact simple-Lie-algebra gauge groups

Fix throughout a compact connected Lie group G whose Lie algebra $\mathfrak{g}$ is simple. The global form of G is not fixed: it may be simply connected, adjoint, or an intermediate quotient. Let

$$\pi : \tilde{G} \longrightarrow G, \quad \Gamma = \text{Ker } \pi \cong \pi_1(G),$$

be the universal covering homomorphism. Then Γ is finite and central. Fix an Ad-invariant inner product $(\cdot, \cdot)_{\mathfrak{g}}$, normalized so that the adjoint quadratic Casimir is $C_A > 0$. Fix once and for all a faithful finite-dimensional unitary representation

$$\rho : G \longrightarrow U(d_\rho).$$

There is a unique $c_\rho > 0$ such that

$$-\text{Tr}(d\rho(X)d\rho(Y)) = c_\rho(X, Y)_{\mathfrak{g}}.$$

Define the normalized plaquette defect

$$\ell_\rho(g) = c_\rho^{-1} \text{Re Tr}(I_{d_\rho} - \rho(g)). \tag{14}$$

Lemma 7.1 (Plaquette-defect normalization). *The function ℓ_ρ is continuous, nonnegative, conjugation invariant, and satisfies*

$$\ell_\rho(g) = 0 \iff g = e.$$

For $X \rightarrow 0$ in $\mathfrak{g}$,

$$\ell_\rho(\exp X) = \frac{1}{2}(X, X)_{\mathfrak{g}} + O(\|X\|^4).$$

Proof. Every eigenvalue of the unitary matrix $\rho(g)$ has the form $e^{i\theta_j}$, so

$$\operatorname{Re} \operatorname{Tr}(I - \rho(g)) = \sum_j (1 - \cos \theta_j) \geq 0.$$

It vanishes exactly when every eigenvalue equals one, hence when $\rho(g) = I$. Faithfulness gives $g = e$. Conjugation invariance and continuity follow from the trace. Since $\mathfrak{g}$ is semisimple, $\operatorname{Tr}(d\rho(X)) = 0$. Expanding the exponential and taking real parts gives

$$\operatorname{Re} \operatorname{Tr}(I - e^{d\rho(X)}) = -\frac{1}{2} \operatorname{Tr}(d\rho(X)^2) + O(\|X\|^4) = \frac{c_\rho}{2}(X, X)_{\mathfrak{g}} + O(\|X\|^4).$$

□

7.2 Physical scale and the diagonal regulator sequence

Fix a reference inverse length $\Lambda_* > 0$. All dimensionful constants are measured relative to Λ_* ; in particular a proposed physical mass m means m/Λ_* is held fixed as the regulator is removed. For integers $n \geq n_0$, define

$$\lambda_n = (a_n, a_{t,n}, L_n, a_{\tau,n}, N_{\tau,n}) = \left(\frac{2^{-n}}{\Lambda_*}, \frac{2^{-n}}{\Lambda_*}, 2^{2n}, \frac{2^{-2n}}{\Lambda_*}, 2^n \right). \quad (15)$$

Here L_n is the number of sites in each spatial direction. Thus

$$a_n \Lambda_* = a_{t,n} \Lambda_* = 2^{-n}, \quad a_{\tau,n} \Lambda_* = 2^{-2n}, \quad (16)$$

$$R_n := L_n a_n = \frac{2^n}{\Lambda_*} \longrightarrow \infty, \quad \beta_{\tau,n} := N_{\tau,n} a_{\tau,n} = \frac{2^{-n}}{\Lambda_*} \longrightarrow 0.$$

This is one diagonal family, not a collection of unspecified iterated limits.

7.3 A precise two-loop reference bare-coupling sequence

Set

$$b_0 = \frac{11C_A}{3(4\pi)^2}, \quad b_1 = \frac{34C_A^2}{3(4\pi)^4}. \quad (17)$$

For $x > 0$, define

$$\Phi_G(x) = \exp\left(-\frac{1}{2b_0x}\right) (b_0x)^{-b_1/(2b_0^2)}. \quad (18)$$

Choose n_0 large enough that $2^{-n} < \Phi_G(b_0/(2b_1))$ for every $n \geq n_0$, and define $g_n > 0$ to be the unique solution in

$$0 < g_n^2 < \frac{b_0}{2b_1}$$

of

$$a_n \Lambda_* = 2^{-n} = \Phi_G(g_n^2). \quad (19)$$

Proposition 7.2 (Existence and ultraviolet behavior of the reference sequence). *The sequence g_n in equation (19) is well defined, satisfies $g_n \rightarrow 0$, and obeys*

$$g_n^2 = \frac{1}{2b_0n \log 2} \left(1 + O\left(\frac{\log n}{n}\right) \right). \quad (20)$$

Proof. A direct differentiation gives

$$\frac{d}{dx} \log \Phi_G(x) = \frac{b_0 - b_1 x}{2b_0^2 x^2}.$$

Hence Φ_G is strictly increasing on $(0, b_0/b_1)$, and $\Phi_G(x) \rightarrow 0$ as $x \downarrow 0$. The choice of n_0 gives existence and uniqueness in the declared interval. Taking logarithms of equation (19) yields

$$n \log 2 = \frac{1}{2b_0 g_n^2} + \frac{b_1}{2b_0^2} \log(b_0 g_n^2).$$

The first term forces $g_n \rightarrow 0$. Substitution of the leading estimate back into the logarithmic term gives equation (20). $\square$

Warning 7.3 (What theorem 7.2 does not prove). The relation equation (19) freezes a numerical sequence modeled on the universal perturbative coefficients [13–16]. It does not prove that the nonperturbative lattice measures converge, that the resulting continuum theory is asymptotically free, or that Λ_* equals a dynamically generated mass scale. Those would be separate theorems.

7.4 Frozen auxiliary fast-sector scaling

The completion-band regulator must be frozen before any spectral estimate is attempted. Fix constants

$$0 < \sigma_H < 1, \quad \kappa_* > 0,$$

and set

$$\kappa_{H,n} = \kappa_* \Lambda_* N_{\tau,n}^{-(1-\sigma_H)}, \quad c_{\tau,n}^{\text{kin}} = \frac{g_n^2 a_n}{2a_{\tau,n}^2 N_{\tau,n}}, \quad \alpha_n^{\text{mix}} = \frac{a_n}{2g_n^2 N_{\tau,n}}. \quad (21)$$

The coefficient $c_{\tau,n}^{\text{kin}}$ is the exact auxiliary-link Laplace coefficient inherited from $\mathcal{L}_n$. The coefficient α_n^{mix} is the tangent coefficient of the normalized mixed-curvature potential after the expansion $\ell_\rho(e^X) = \frac{1}{2}\|X\|^2 + O(\|X\|^4)$. The closure coefficient tends to zero as a bare local coupling, while its coherent mean-mode Hessian is amplified by the product around the $N_{\tau,n}$ -site auxiliary cycle.

Proposition 7.4 (Algebraic fast-scale window). *For the diagonal and coupling sequences in (15) and (19),*

$$c_{\tau,n}^{\text{kin}} = \frac{g_n^2}{2} 2^{2n} \Lambda_*, \quad (22)$$

$$\gamma_{\text{rel},n}^{\text{lin}} := \frac{16\alpha_n^{\text{mix}}}{\beta_{\tau,n}^2} = \frac{8\Lambda_*}{g_n^2}, \quad (23)$$

$$\gamma_{H,n}^{\text{lin}} := \frac{1}{2} \kappa_{H,n} N_{\tau,n} = \frac{\kappa_* \Lambda_*}{2} N_{\tau,n}^{\sigma_H}. \quad (24)$$

Consequently

$$c_{\tau,n}^{\text{kin}} \longrightarrow \infty, \quad \gamma_{\text{rel},n}^{\text{lin}} \longrightarrow \infty, \quad \gamma_{H,n}^{\text{lin}} \longrightarrow \infty,$$

whereas

$$a_{t,n} \kappa_{H,n} = \kappa_* 2^{-n(2-\sigma_H)} \longrightarrow 0. \quad (25)$$

Thus the frozen sequence has a nonempty linear scaling window in which the auxiliary tangent stiffness diverges but the one-step bare closure coefficient vanishes.

Proof. Insert $a_n = a_{t,n} = 2^{-n}\Lambda_*^{-1}$, $a_{\tau,n} = 2^{-2n}\Lambda_*^{-1}$, $N_{\tau,n} = 2^n$, and $\beta_{\tau,n} = a_n$ into equation (21). Since $N_{\tau,n}a_n = \Lambda_*^{-1}$,

$$\frac{16\alpha_n^{\text{mix}}}{\beta_{\tau,n}^2} = \frac{8}{g_n^2 N_{\tau,n} a_n} = \frac{8\Lambda_*}{g_n^2}.$$

The other identities are immediate. Finally $g_n \rightarrow 0$ by theorem 7.2, so all three positive fast scales diverge. The limit (25) follows directly from the chosen exponent $0 < \sigma_H < 1$. $\square$

Proof Audit 7.5 (What the algebraic window does not prove). The divergence in equations (23) and (24) is a tangent-space statement. It does not prove a nonlinear spectral band, uniform control of gauge charts, absence of remote wells, smallness of induced four-dimensional operators, or route independence. Those obligations are isolated in theorems 15.4, 16.4, 19.4 and 19.5.

7.5 Spatial boundary and flux sectors

Let T^3 denote the spatial three-torus. Since G is connected and $\pi_2(G) = 0$, obstruction theory on the three-dimensional CW complex T^3 gives the magnetic bundle labels [17, 44]

$$m \in H^2(T^3; \Gamma). \quad (26)$$

For each m , choose once and for all a lattice transition cocycle representing the corresponding principal G -bundle P_m . The finite collection of central boundary automorphisms preserving that cocycle is denoted $\mathcal{C}_m$. It is finite abelian. Its character labels

$$\chi \in \widehat{\mathcal{C}_m} := \text{Hom}(\mathcal{C}_m, U(1)) \quad (27)$$

are the center/electric-flux sectors. For the untwisted bundle, $\mathcal{C}_0$ contains the familiar center transformations around the three spatial cycles; when the global form of G has trivial center, this factor can be trivial.

Proposition 7.6 (Complete topological and central boundary bookkeeping). *At fixed n , the boundary data relevant here decompose into the disjoint magnetic labels $m \in H^2(T^3; \Gamma)$. In each magnetic sector the unitary action of $\mathcal{C}_m$ decomposes the gauge-invariant slice Hilbert space as*

$$\mathcal{H}_{n,m}^{\text{gauge}} = \bigoplus_{\chi \in \widehat{\mathcal{C}_m}} \mathcal{H}_{n,m,\chi}. \quad (28)$$

Every local plaquette action and every heat-kernel transfer operator constructed below preserves m and commutes with $\mathcal{C}_m$, hence reduces each summand.

Proof. For a connected group, BG is simply connected, $\pi_2(BG) = \pi_1(G) = \Gamma$, and $\pi_3(BG) = \pi_2(G) = 0$. The three-stage Postnikov truncation of BG is therefore $K(\Gamma, 2)$. Maps from the three-complex T^3 to BG , equivalently isomorphism classes of principal G -bundles, are classified by $H^2(T^3; \Gamma)$. Once a representative cocycle is fixed, central boundary automorphisms act unitarily on square-integrable gauge-invariant functions. The ordinary character decomposition for the finite abelian group $\mathcal{C}_m$ gives equation (28). Plaquette holonomies transform by conjugation and heat kernels are central, so both constructions commute with these automorphisms and cannot change the fixed transition cocycle. $\square$

Definition 7.7 (Frozen physical candidate sector). The main proof is carried out in the untwisted, center-neutral spatial sector

$$m = 0, \quad \chi = 1,$$

and later in the auxiliary-translation-neutral subspace of that sector. This choice is part of the regulator definition. Failure of a required estimate in this vacuum sector already prevents a theorem asserted for every allowed group and every physical sector.

7.6 Exact order of operations and limits

The frozen order is:

- (1) Fix G, ρ, Λ_* , the cocycle representatives, and the sector $(m, \chi) = (0, 1)$.
- (2) At each fixed n , construct the compact slice theory, normalize its transfer operator, and construct its two-sided infinite outer-time vacuum state. If a finite outer-time cylinder is used as an approximation, its length tends to infinity at fixed n before any regulator limit.
- (3) Establish estimates uniform in n in physical units.
- (4) Only after step (3), take the single diagonal limit $n \rightarrow \infty$, which simultaneously sends $a_n, a_{t,n}, a_{\tau,n} \rightarrow 0$, $L_n a_n \rightarrow \infty$, $N_{\tau,n} \rightarrow \infty$, and $N_{\tau,n} a_{\tau,n} \rightarrow 0$.

The hard-window version of step (3) fails in [theorem 1.1](#). The completion-band redesign replaces that failed estimate, but its nonlinear, RG, and continuum estimates remain program requirements; consequently step (4) is not performed in this manuscript.

8 Finite-regulator heat-kernel transfer theory

8.1 Slice lattice, variables, and gauge action

For fixed n , let

$$\Sigma_n = (\mathbb{Z}/L_n\mathbb{Z})^3, \quad C_n = \mathbb{Z}/N_{\tau,n}\mathbb{Z}.$$

A vertex of one outer-time slice is $(x, \tau) \in \Sigma_n \times C_n$. There are spatial links

$$U_i(x, \tau) \in G, \quad i = 1, 2, 3,$$

and auxiliary links

$$V(x, \tau) \in G.$$

The untwisted slice configuration space is the compact product

$$\mathcal{X}_n = G^{3L_n^3 N_{\tau,n}} \times G^{L_n^3 N_{\tau,n}}, \quad (29)$$

with normalized product Haar measure μ_n . The slice gauge group

$$\mathcal{G}_n = G^{L_n^3 N_{\tau,n}}$$

acts by

$$U_i^g(x, \tau) = g(x, \tau) U_i(x, \tau) g(x + \hat{i}, \tau)^{-1}, \quad (30)$$

$$V^g(x, \tau) = g(x, \tau) V(x, \tau) g(x, \tau + 1)^{-1}. \quad (31)$$

Indices are periodic.

The spatial and mixed plaquettes are

$$W_{ij}(x, \tau) := U_i(x, \tau) U_j(x + \hat{i}, \tau) U_i(x + \hat{j}, \tau)^{-1} U_j(x, \tau)^{-1}, \quad (32)$$

$$W_{i\tau}(x, \tau) := V(x, \tau) U_i(x, \tau + 1) V(x + \hat{i}, \tau)^{-1} U_i(x, \tau)^{-1}. \quad (33)$$

Both transform by conjugation at (x, τ) .

8.2 The frozen kinetic and potential coefficients

Let Δ_ℓ denote the nonnegative Laplace–Beltrami operator on the copy of G attached to link ℓ , for the bi-invariant metric induced by $(\cdot, \cdot)_\mathfrak{g}$. Define on $C^\infty(\mathcal{X}_n)$

$$\mathcal{L}_n = \frac{g_n^2}{2a_n N_{\tau,n}} \sum_{x,\tau,i} \Delta_{U_i(x,\tau)} + \frac{g_n^2 a_n}{2a_{\tau,n}^2 N_{\tau,n}} \sum_{x,\tau} \Delta_{V(x,\tau)}. \quad (34)$$

The within-slice potential is

$$\begin{aligned} \mathcal{V}_n(U, V) &= \frac{1}{g_n^2 a_n N_{\tau,n}} \sum_{x,\tau} \sum_{1 \leq i < j \leq 3} \ell_\rho(W_{ij}(x, \tau)) \\ &\quad + \frac{a_n}{g_n^2 a_{\tau,n}^2 N_{\tau,n}} \sum_{x,\tau,i} \ell_\rho(W_{i\tau}(x, \tau)). \end{aligned} \quad (35)$$

The mixed-energy function and multiplication operator are

$$\mathcal{K}_n(U, V) := \frac{a_n}{a_{\tau,n}^2 N_{\tau,n}} \sum_{x,\tau,i} \ell_\rho(W_{i\tau}(x, \tau)), \quad (36)$$

$$(K_n f)(U, V) := \mathcal{K}_n(U, V) f(U, V). \quad (37)$$

Thus the mixed part of $\mathcal{V}_n$ is $g_n^{-2} \mathcal{K}_n$. All coefficients are now fixed functions of n ; no adjustable anisotropy remains.

Proposition 8.1 (Basic operator properties). *The operator $\mathcal{L}_n$ is essentially self-adjoint and nonnegative on $C^\infty(\mathcal{X}_n)$, has compact resolvent, and commutes with the gauge action, spatial translations, cubic rotations, and cyclic auxiliary translations. The functions $\mathcal{V}_n$ and $\mathcal{K}_n$ are continuous, bounded, nonnegative, gauge invariant, and invariant under the same lattice symmetries. Consequently K_n is bounded, nonnegative, and self-adjoint.*

Proof. The sum in equation (34) is a strictly elliptic Laplace-type operator on the compact product manifold $\mathcal{X}_n$. Standard elliptic theory gives essential self-adjointness on smooth functions, nonnegativity, and compact resolvent. Bi-invariance of each link metric makes the Laplacian commute with left and right translations, hence with the gauge action. Translation-invariance of the coefficients gives the lattice symmetries. The remaining claims follow from theorem 7.1, compactness of $\mathcal{X}_n$, and conjugation covariance of plaquette holonomies. $\square$

8.3 Gauge-invariant Hilbert space and heat kernel

Set

$$\mathcal{H}_n^{\text{all}} = L^2(\mathcal{X}_n, \mu_n).$$

The gauge-invariant projection and slice Hilbert space are

$$(\Pi_{\mathcal{G},n} f)(X) = \int_{\mathcal{G}_n} f(g^{-1} \cdot X) \, dg, \quad \mathcal{H}_n^{\text{sl}} = \text{Ran } \Pi_{\mathcal{G},n}. \quad (38)$$

The heat semigroup on the unreduced space is

$$\mathbf{P}_n^{\text{all}} = \exp(-a_{t,n} \mathcal{L}_n).$$

Since $\mathcal{L}_n$ commutes with $\Pi_{\mathcal{G},n}$, define

$$\mathbf{P}_n = \mathbf{P}_n^{\text{all}}|_{\mathcal{H}_n^{\text{sl}}}. \quad (39)$$

For a spatial link set

$$s_{s,n} = a_{t,n} \frac{g_n^2}{2a_n N_{\tau,n}},$$

and for an auxiliary link set

$$s_{\tau,n} = a_{t,n} \frac{g_n^2 a_n}{2a_{\tau,n}^2 N_{\tau,n}}.$$

Let h_s be the central heat kernel on G for $\exp(-s\Delta)$. If E_n denotes all spatial and auxiliary links in the slice and $s_{\ell,n}$ is the appropriate one of these two times, the unreduced kernel is

$$\mathbf{p}_n^0(X, Y) = \prod_{\ell \in E_n} h_{s_{\ell,n}}(X_\ell Y_\ell^{-1}). \quad (40)$$

Proposition 8.2 (Gauge-averaged temporal-link kernel). *The restriction $\mathbf{P}_n$ has the separately gauge-invariant smooth kernel*

$$\mathbf{p}_n^G(X, Y) = \int_{\mathcal{G}_n} \prod_{\ell \in E_n} h_{s_{\ell,n}}(X_\ell g_{t(\ell)} Y_\ell^{-1} g_{s(\ell)}^{-1}) \prod_v dg_v. \quad (41)$$

The integration variables g_v can be read as outer-time links between two adjacent slices, and every factor is the heat-kernel weight of the corresponding outer-time plaquette. The kernel is strictly positive.

Proof. The product formula equation (40) follows from commutativity of the link Laplacians. Gauge averaging in the second variable gives

$$\int_{\mathcal{G}_n} \mathbf{p}_n^0(X, g \cdot Y) dg.$$

For a link $s(\ell) \rightarrow t(\ell)$,

$$(g \cdot Y)_\ell = g_{s(\ell)} Y_\ell g_{t(\ell)}^{-1},$$

which gives equation (41). Each compact-group heat kernel is smooth and strictly positive for positive time, and averaging preserves strict positivity. Centrality gives separate gauge invariance and the plaquette interpretation. $\square$

8.4 Transfer operator, vacuum, and Hamiltonian

Let

$$M_n = \exp(-a_{t,n} \mathcal{V}_n / 2) \quad (42)$$

act by multiplication. Since $\mathcal{V}_n$ is bounded, M_n and M_n^{-1} are bounded. Define

$$T_n^{(0)} = M_n \mathbf{P}_n M_n, \quad T_n = \frac{T_n^{(0)}}{\|T_n^{(0)}\|}. \quad (43)$$

Theorem 8.3 (Finite-regulator transfer operator). *On $\mathcal{H}_n^{\text{sl}}$, the operator T_n is self-adjoint, positive as an operator, positivity preserving, injective, trace class, and a contraction. It is positivity improving on the cone of nonnegative gauge-invariant functions. Its spectral radius is the simple eigenvalue 1, with a unique normalized eigenfunction Ω_n that is smooth and strictly positive at every point of $\mathcal{X}_n$.*

Proof. Self-adjointness and operator positivity follow from

$$\langle f, T_n^{(0)} f \rangle = \langle M_n f, P_n M_n f \rangle \geq 0.$$

The heat semigroup is trace class on the compact product and restriction to a reducing subspace preserves that property; multiplication by bounded operators preserves trace class. If $T_n^{(0)} f = 0$, then the displayed quadratic form is zero. Spectral calculus for $\mathcal{L}_n$ gives $\text{Ker } P_n = \{0\}$ and $\langle h, P_n h \rangle > 0$ for every $h \neq 0$, so $M_n f = 0$, and invertibility of M_n yields $f = 0$. The kernel from theorem 8.2, multiplied by the two strictly positive potential factors, is strictly positive. Therefore a nonzero nonnegative function is mapped to a pointwise positive function. The Jentzsch–Perron theorem for compact positivity-improving integral operators gives a simple strictly positive top eigenvector; this is the transfer-matrix mechanism used in rigorous lattice gauge theory [20, 21]. Division by the norm makes its eigenvalue one. Smoothness follows from heat-kernel smoothing, and strict positivity is pointwise. Compactness of $\mathcal{X}_n$ also gives $\min \Omega_n > 0$. $\square$

Definition 8.4 (Physical Hamiltonian and domains). Let $E_n(\cdot)$ be the spectral measure of T_n . Define

$$H_n = -a_{t,n}^{-1} \log T_n \quad (44)$$

with

$$\text{Dom}(H_n) = \left\{ f \in \mathcal{H}_n^{\text{sl}} : \int_{(0,1]} a_{t,n}^{-2} |\log t|^2 \, d\langle f, E_n(t) f \rangle < \infty \right\}. \quad (45)$$

Its closed quadratic form q_n has domain

$$\text{Dom}(q_n) = \left\{ f : \int_{(0,1]} a_{t,n}^{-1} |\log t| \, d\langle f, E_n(t) f \rangle < \infty \right\}. \quad (46)$$

Proposition 8.5 (Self-adjointness and fixed-regulator gap). *The operator H_n is densely defined, nonnegative, and self-adjoint; $H_n \Omega_n = 0$. At each fixed n there is a number $\Delta_n > 0$ such that*

$$H_n \geq \Delta_n I \quad \text{on } \Omega_n^\perp. \quad (47)$$

No lower bound on $\inf_n \Delta_n$ follows from this proposition.

Proof. Injectivity of T_n means its spectral projection at zero vanishes. The measurable functional calculus therefore defines $-\log T_n$ on the dense domain equation (45); it is self-adjoint and nonnegative because $0 < T_n \leq I$. Since $T_n \Omega_n = \Omega_n$, the vacuum has energy zero. Compactness and simplicity of the top eigenvalue imply that the remainder of $\text{Spec}(T_n)$ lies in $[0, r_n]$ for some $r_n < 1$. Thus $\Delta_n = -a_{t,n}^{-1} \log r_n > 0$. The argument contains no estimate uniform in n . $\square$

8.5 Physical normalization on the zero-mixed stratum

Suppose $V(x, \tau) = I$ and $U_i(x, \tau) = U_i(x)$ are independent of τ . Then all mixed plaquettes are the identity and

$$\mathcal{V}_n(U, I) = \frac{1}{g_n^2 a_n} \sum_x \sum_{i < j} \ell_\rho(W_{ij}(x)). \quad (48)$$

Thus the declared $1/N_{\tau,n}$ normalization makes the surviving spatial magnetic term exactly the standard Kogut–Susskind normalization associated with ℓ_ρ , rather than $N_{\tau,n}$ copies of it [12]. This identity is a finite-lattice normalization statement only.

9 Vacuum process, auxiliary neutrality, and common-space mixed energy

9.1 Ground-state transform

Define the vacuum probability measure

$$d\nu_n(X) = \Omega_n(X)^2 d\mu_n(X) \quad (49)$$

and the unitary

$$U_{\Omega,n} : L^2(\mathcal{X}_n, \nu_n)^{\mathcal{G}_n} \longrightarrow \mathcal{H}_n^{\text{sl}}, \quad U_{\Omega,n}f = \Omega_n f.$$

The transformed transfer operator

$$\hat{T}_n = U_{\Omega,n}^{-1} T_n U_{\Omega,n} \quad (50)$$

has kernel relative to ν_n

$$\hat{t}_n(X, Y) = \frac{t_n(X, Y)}{\Omega_n(X)\Omega_n(Y)}, \quad (51)$$

where t_n is the integral kernel of T_n .

Proposition 9.1 (Reversible Markov operator). *The operator $\hat{T}_n$ is self-adjoint, positivity improving, and satisfies*

$$\hat{T}_n 1 = 1.$$

Equivalently, $\hat{t}_n$ is a strictly positive symmetric Markov kernel relative to ν_n .

Proof. Unitarity transfers self-adjointness and positivity. The eigenvalue equation $T_n \Omega_n = \Omega_n$ gives $\hat{T}_n 1 = 1$. Substitution into the integral formula gives

$$\int \hat{t}_n(X, Y) d\nu_n(Y) = 1.$$

Symmetry follows from symmetry of t_n , and strict positivity follows from theorem 8.3. $\square$

Let $\mathbb{P}_n^{\text{vac}}$ be the unique two-sided stationary Markov measure on $\mathcal{X}_n^{\mathbb{Z}}$ with one-time law ν_n and transition operator $\hat{T}_n$. Its existence follows from the Kolmogorov extension theorem.

Theorem 9.2 (Finite-regulator reflection positivity and OS reconstruction). *Let θ reflect outer time, $(\theta X)_k = X_{-k}$, and put $\Theta F = \overline{F} \circ \theta$. If F is a bounded cylinder function depending only on times $0, 1, \dots, r$, then*

$$\int (\Theta F) F d\mathbb{P}_n^{\text{vac}} = \int |\mathbb{E}_n^{\text{vac}}[F \mid X_0]|^2 d\nu_n \geq 0. \quad (52)$$

In the sense of the Osterwalder–Schrader reconstruction [18, 19], the resulting OS Hilbert space is naturally unitarily equivalent to $L^2(\nu_n)^{\mathcal{G}_n}$, hence through $U_{\Omega,n}$ to $\mathcal{H}_n^{\text{sl}}$. One-step OS time translation is $\hat{T}_n$, and its physical generator is unitarily equivalent to H_n .

Proof. Conditioned on X_0 , the past and future of a two-sided Markov chain are independent. Reversibility identifies the conditional law of the reflected past with that of the future. Therefore the conditional expectation of $(\Theta F)F$ given X_0 is the modulus square in equation (52). Quotienting positive-time cylinders by the null space of this form identifies a cylinder with its conditional expectation given X_0 . Every bounded gauge-invariant function of X_0 occurs, and these functions are dense in $L^2(\nu_n)^{\mathcal{G}_n}$, proving the Hilbert-space identification. The Markov property identifies one-step translation with $\hat{T}_n$; equations (44) and (50) then identify its generator. $\square$

Remark 9.3 (Periodic loops are not the vacuum). A finite periodic outer-time loop with weight $\text{Tr}(T_n^M \cdot)$ is a trace state. It is not the pure vacuum state at finite M . Since the top eigenvalue of T_n is simple, normalized periodic expectations converge to vacuum expectations as $M \rightarrow \infty$ at fixed n . The infinite two-sided process above constructs the vacuum directly and avoids conflating these states.

9.2 Auxiliary translation and the neutral candidate sector

Let $R_{\tau,n}$ shift every link variable by one unit in the auxiliary coordinate. It is unitary and

$$[R_{\tau,n}, \mathcal{L}_n] = [R_{\tau,n}, \mathcal{V}_n] = [R_{\tau,n}, T_n] = [R_{\tau,n}, K_n] = 0. \quad (53)$$

Define

$$P_{0,n} = \frac{1}{N_{\tau,n}} \sum_{j=0}^{N_{\tau,n}-1} R_{\tau,n}^j, \quad \mathcal{N}_n = P_{0,n} \mathcal{H}_n^{\text{sl}}. \quad (54)$$

Proposition 9.4 (Neutral reducing sector). *The space $\mathcal{N}_n$ reduces T_n , H_n , and K_n . Moreover $\Omega_n \in \mathcal{N}_n$.*

Proof. The commutators in equation (53) imply reduction. Since $R_{\tau,n}\Omega_n$ is another strictly positive normalized eigenvector of T_n at eigenvalue one, simplicity from theorem 8.3 gives $R_{\tau,n}\Omega_n = \Omega_n$. $\square$

The label “physical candidate” is intentionally provisional. Auxiliary-translation neutrality does not imply independence of auxiliary holonomy or of relative auxiliary configurations. Proving that all such observables disappear in a continuum limit is the separate auxiliary-invisibility target, which is not reached here.

9.3 The multiplication spectral window

On $\mathcal{N}_n$, define for $\varepsilon > 0$

$$P_{\varepsilon,n} = \mathbf{1}_{[0,\varepsilon]}(K_n)|_{\mathcal{N}_n}, \quad Q_{\varepsilon,n} = I - P_{\varepsilon,n}. \quad (55)$$

Because K_n is multiplication by $\mathcal{K}_n$, $P_{\varepsilon,n}$ is multiplication by the indicator of the gauge-invariant set

$$\mathcal{A}_{\varepsilon,n} = \{X \in \mathcal{X}_n : \mathcal{K}_n(X) \leq \varepsilon\}.$$

The projection need not preserve $\Omega_n^\perp$, because $K_n\Omega_n$ is generally not proportional to Ω_n . This is the reason for using the vacuum-corrected gap operator in section 12.

10 Holonomy stratification, zero sets, and finite-block diagnostics

10.1 Mixed plaquettes as covariant differences

For a spatial link (x, i) at auxiliary step τ , define

$$\nabla_\tau^{\text{cov}} U_i(x, \tau) = \frac{1}{a_{\tau,n}} \left[\rho(V(x, \tau) U_i(x, \tau + 1)) - \rho(U_i(x, \tau) V(x + \hat{i}, \tau)) \right]. \quad (56)$$

Lemma 10.1 (Exact square identity). *For every mixed plaquette,*

$$\ell_\rho(W_{i\tau}(x, \tau)) = \frac{a_{\tau,n}^2}{2c_\rho} \|\nabla_\tau^{\text{cov}} U_i(x, \tau)\|_{\text{HS}}^2. \quad (57)$$

Proof. Put $A = \rho(V(x, \tau)U_i(x, \tau + 1))$ and $B = \rho(U_i(x, \tau)V(x + \hat{i}, \tau))$. Both are unitary, so

$$\|A - B\|_{\text{HS}}^2 = 2d_\rho - 2\text{Re Tr}(A^*B).$$

By cyclicity of the trace, $\text{Re Tr}(A^*B) = \text{Re Tr}(\rho(W_{i\tau}^{-1})) = \text{Re Tr}(\rho(W_{i\tau}))$. Apply equation (14) and divide by $a_{\tau,n}^2$. $\square$

This pointwise identity does not identify the spectrum of the finite-dimensional difference with the spectrum of the multiplication operator K_n or the Hamiltonian H_n .

10.2 Auxiliary axial gauge and holonomy strata

At each spatial site define the auxiliary holonomy

$$H_x = V(x, 0)V(x, 1) \cdots V(x, N_{\tau,n} - 1). \quad (58)$$

Lemma 10.2 (Auxiliary axial gauge). *Every configuration is gauge equivalent to one satisfying*

$$V(x, \tau) = e \quad (0 \leq \tau < N_{\tau,n} - 1), \quad V(x, N_{\tau,n} - 1) = H_x. \quad (59)$$

Residual gauge transformations are independent of τ and act by

$$U_i(x, \tau) \mapsto h_x U_i(x, \tau) h_{x+\hat{i}}^{-1}, \quad H_x \mapsto h_x H_x h_x^{-1}.$$

Proof. Set $g(x, 0) = e$ and recursively

$$g(x, \tau + 1) = g(x, \tau)V(x, \tau) \quad (0 \leq \tau < N_{\tau,n} - 1).$$

Substitution into equation (31) sets the first $N_{\tau,n} - 1$ auxiliary links to the identity and leaves their ordered product on the closing link. A transformation preserving this gauge must have $h(x, \tau + 1) = h(x, \tau)$, hence is independent of τ . $\square$

Theorem 10.3 (Holonomy stratification of the zero-mixed set). *In auxiliary axial gauge, the conditions*

$$W_{i\tau}(x, \tau) = e \quad \text{for every } x, i, \tau \quad (60)$$

are equivalent to the existence of spatial links $U_i(x)$, independent of τ , and holonomies H_x satisfying

$$H_x U_i(x) = U_i(x) H_{x+\hat{i}}. \quad (61)$$

Modulo residual gauge transformations, the zero set is stratified by the simultaneous conjugacy orbits of the holonomy field $\{H_x\}$ subject to equation (61). The stratum $H_x = e$ for all x is exactly the ordinary three-dimensional spatial lattice-gauge configuration quotient at one outer-time slice.

Proof. For $0 \leq \tau < N_{\tau,n} - 1$, equations (33) and (59) reduce the condition $W_{i\tau} = e$ to

$$U_i(x, \tau + 1) = U_i(x, \tau).$$

Thus U_i is independent of τ . The closing mixed plaquette is

$$H_x U_i(x) H_{x+\hat{i}}^{-1} U_i(x)^{-1},$$

so it equals the identity exactly when equation (61) holds. Reversing these steps proves the converse. The residual action is given in theorem 10.2. If every $H_x = e$, only the τ -independent spatial links and ordinary spatial gauge transformations remain. $\square$

Corollary 10.4 (Ordinary Yang–Mills stratum). *On the trivial-holonomy part of the zero-mixed set, $\mathcal{H}_n = 0$ and the potential reduces to equation (48). Hence every positive low window contains neighborhoods of configurations whose auxiliary reduction is ordinary lattice Yang–Mills. The auxiliary term cannot supply a strictly positive pointwise mass floor on that stratum.*

Proof. The mixed plaquettes are the identity, so every term in equation (36) vanishes. The physical potential identity was proved in equation (48). Continuity places neighborhoods of the zero set in every positive-width low window. $\square$

10.3 Holonomy-twisted fiber spectrum

Fix a spatial link $e = (x, i)$ and define the unitary map on $\text{End}(\mathbb{C}^{d_\rho})$

$$\mathcal{H}_e(B) = \rho(H_x) B \rho(H_{x+i})^{-1}. \quad (62)$$

For a sequence $A = (A_0, \dots, A_{N-1})$, $N = N_{\tau,n}$, define

$$(D_{\tau, \mathcal{H}_e} A)_r = a_{\tau,n}^{-1} (A_{r+1} - A_r), \quad 0 \leq r < N-1, \quad (63)$$

$$(D_{\tau, \mathcal{H}_e} A)_{N-1} = a_{\tau,n}^{-1} (\mathcal{H}_e A_0 - A_{N-1}). \quad (64)$$

Theorem 10.5 (Exact holonomy-twisted spectrum). *Let $e^{i\theta_j}$, $-\pi < \theta_j \leq \pi$, be the eigenvalues of $\mathcal{H}_e$, counted with multiplicity. Then*

$$\text{Spec}(D_{\tau, \mathcal{H}_e}^* D_{\tau, \mathcal{H}_e}) = \left\{ \frac{4}{a_{\tau,n}^2} \sin^2 \left(\frac{2\pi k + \theta_j}{2N} \right) : 0 \leq k < N, j \right\}. \quad (65)$$

The kernel consists of constant sequences $A_r = A_0$ satisfying $\mathcal{H}_e A_0 = A_0$.

Proof. Choose an orthonormal eigenbasis b_j for the unitary $\mathcal{H}_e$. The vectors

$$A_{k,j}(r) = N^{-1/2} \exp \left(\frac{i(2\pi k + \theta_j)r}{N} \right) b_j$$

satisfy the twisted closing condition and diagonalize $D_{\tau, \mathcal{H}_e}$. The multiplier is

$$a_{\tau,n}^{-1} \left(\exp \left(\frac{i(2\pi k + \theta_j)}{N} \right) - 1 \right),$$

whose squared modulus is equation (65). It vanishes exactly for $k = 0$, $\theta_j = 0$, giving the stated kernel. $\square$

Corollary 10.6 (No compactification shortcut). *After removal of the exact kernel, no positive lower bound follows from compactness of the auxiliary cycle alone. Holonomy eigenphases $\theta_j \rightarrow 0$ produce positive eigenvalues in equation (65) tending to zero.*

Proof. Take $k = 0$ and nonzero $\theta_j \rightarrow 0$. The corresponding eigenvalue is asymptotic to $\theta_j^2 / (N^2 a_{\tau,n}^2)$. $\square$

10.4 Local zero spaces versus quantum ground spaces

A finite-block detectability argument normally begins with nonzero local ground-space projections. The continuous-group multiplication terms used here do not have such eigenspaces.

Proposition 10.7 (Classical local zeros have no nonzero L^2 ground vector). *Let p be any plaquette in a finite block containing at least one unconstrained link variable, and let*

$$V_p = M_{\ell_\rho(W_p)}$$

act on the corresponding product-Haar L^2 space or its gauge-invariant subspace. Then

$$\text{Ker } V_p = \{0\}, \quad \inf \text{Spec}(V_p) = 0. \quad (66)$$

The same conclusion holds for a finite nonempty sum of nonnegative plaquette multiplication terms.

Proof. Fix all plaquette links except one. As that link ranges over G , the plaquette holonomy ranges over G by a left-right translate. The condition $W_p = e$ therefore fixes that link to a single group element. Since G has positive dimension, a singleton has Haar measure zero. Fubini's theorem shows that the full zero set $\{W_p = e\}$ has product-Haar measure zero. By [theorem 7.1](#), a function in $\text{Ker } V_p$ must be supported on this null set, so it is zero in L^2 . On the other hand, every neighborhood of an identity configuration has positive measure and contains values of $\ell_\rho(W_p)$ arbitrarily close to zero; hence zero belongs to the essential range and is the spectral infimum. A sum can vanish only on the zero set of each term and in particular on the null set of one chosen term, so its kernel is also zero and its infimum is zero by neighborhoods of a flat configuration. $\square$

Proof Audit 10.8 (Finite-block consequence). The classical flat configurations in [theorem 10.3](#) can be classified, but they do not furnish nonzero local quantum ground spaces for the multiplication operators. Finite-size and detectability methods [32–35] would first require an additional, rigorously controlled replacement such as spectral flattening, finite representation truncation, or a different local Hamiltonian decomposition. None of those replacements is part of the frozen regulator, and none may be inserted without a comparison theorem. In any event, no finite-block theorem can repair the algebraic failure in [theorem 1.1](#), which occurs on the exact global vacuum line.

10.5 Both low and high windows contain vacuum weight

Lemma 10.9 (Growth and nonconstancy of the mixed energy). *For every n , $\mathcal{K}_n$ is continuous, gauge invariant, and nonconstant, with minimum zero. Moreover*

$$\|K_n\| \geq 2\Lambda_* 2^{2n} \ell_\rho(g_*) \quad (67)$$

for any fixed $g_* \in G \setminus \{e\}$ and a suitable configuration. In particular $\|K_n\| \rightarrow \infty$.

Proof. The all-identity configuration has $\mathcal{K}_n = 0$. Set every auxiliary link to the identity and every spatial link to the identity except one link $U_i(x_0, \tau_0) = g_*$. The two mixed plaquettes adjacent to that link in the auxiliary direction have holonomies conjugate to g_* and g_*^{-1} ; all terms are nonnegative. Since $\ell_\rho(g_*^{-1}) = \ell_\rho(g_*)$,

$$\mathcal{K}_n \geq 2 \frac{a_n}{a_{\tau,n}^2 N_{\tau,n}} \ell_\rho(g_*) = 2\Lambda_* 2^{2n} \ell_\rho(g_*).$$

Continuity and gauge invariance were established in [theorem 8.1](#). $\square$

Theorem 10.10 (Vacuum overlap with every genuine multiplication split). *Fix $\varepsilon > 0$. For every sufficiently large n , let*

$$P = P_{\varepsilon,n}, \quad Q = I - P$$

on $\mathcal{N}_n$. Then

$$P\Omega_n \neq 0, \quad Q\Omega_n \neq 0. \quad (68)$$

More generally, equation (68) holds at any fixed n for which $0 < \varepsilon < \|K_n\|$.

Proof. The set $\{\mathcal{K}_n < \varepsilon\}$ contains an open neighborhood of the identity configuration and therefore has positive product-Haar measure. If $\varepsilon < \|K_n\|$, the set $\{\mathcal{K}_n > \varepsilon\}$ is also a nonempty open set and has positive measure. The continuous vacuum Ω_n is strictly positive everywhere by theorem 8.3; hence its squared norm integrated over either set is strictly positive. Since $\mathcal{K}_n$ and Ω_n are auxiliary-translation invariant, both projected vectors lie in $\mathcal{N}_n$. The final assertion for all sufficiently large n follows from equation (67). $\square$

11 Inner-time coherence, flow interruption, and projection-generated inertia

The symmetry-breaking drafts propose that matter and inertia arise when generative temporal coherence is interrupted or must be retuned. The lattice model supplies an exact gauge-invariant quantity with that interpretation, without adding a new scalar field or breaking the Yang–Mills gauge symmetry [7, 8].

11.1 The covariant interruption defect

Definition 11.1 (Inner-time flow-interruption defect). For a slice configuration $X = (U, V)$, define

$$\mathfrak{I}_{\tau,n}(X) := \mathcal{K}_n(X) = \frac{a_n}{a_{\tau,n}^2 N_{\tau,n}} \sum_{x,\tau,i} \ell_\rho(W_{i\tau}(x, \tau)). \quad (69)$$

Equivalently, by the exact mixed-plaquette identity,

$$\mathfrak{I}_{\tau,n}(X) = \frac{a_n}{2N_{\tau,n}} \sum_{x,\tau,i} \left\| D_{\tau,V}^{\text{cov}} A_{(x,i)}(\tau) \right\|_{\text{HS}}^2. \quad (70)$$

A configuration is *covariantly uninterrupted* when $\mathfrak{I}_{\tau,n} = 0$, and *interrupted* when $\mathfrak{I}_{\tau,n} > 0$.

Proposition 11.2 (Symmetry and zero set of the interruption defect). *The scalar $\mathfrak{I}_{\tau,n}$ is nonnegative and invariant under the slice gauge group, cyclic inner-time translation, spatial translations, and cubic rotations. Moreover,*

$$\mathfrak{I}_{\tau,n}(X) = 0 \iff W_{i\tau}(x, \tau) = e \text{ for every } x, i, \tau. \quad (71)$$

In inner-time axial gauge this is equivalent to covariant constancy of every spatial link history around the inner cycle, with the closing relation determined by the auxiliary holonomy.

Proof. Invariance and nonnegativity follow from theorem 8.1. Every summand in (69) is nonnegative, and $\ell_\rho(g) = 0$ exactly at the identity by theorem 7.1; hence the sum vanishes exactly under (71). The axial-gauge statement is theorem 10.3 restricted to the spatial/inner-time plaquettes. $\square$

The amplitude

$$\Phi_{\tau,n}(X) = \sqrt{\mathfrak{I}_{\tau,n}(X)}$$

may be regarded as a gauge-invariant interruption order parameter. It is the lattice analogue of a norm of a covariant temporal phase gradient. No potential $V(\Phi_{\tau,n})$ is introduced: the exact plaquette action already determines its weight.

11.2 Interruption is not breaking of the gauge symmetry

The frozen transfer theory satisfies

$$[R_{\tau,n}, T_n] = [R_{\tau,n}, K_n] = 0.$$

Thus the action and transfer operator retain the compact inner-time symmetry. A configuration with $\mathfrak{I}_{\tau,n} > 0$ is not an explicit breaking of that global symmetry; it is a local failure of covariant phase matching, comparable to a defect or an excitation in a symmetric theory. Likewise, the local gauge group is a redundancy removed by the physical projection and is never asserted to break. The term *inner-flow interruption* is reserved for two precise phenomena:

- (I1) a nonzero covariant defect $D_{\tau,V}^{\text{cov}} A$, measured by $\mathfrak{I}_{\tau,n}$;
- (I2) a phase slip or nontrivial holonomy obstructing a globally coherent inner history.

Observation filtering by $P_{0,n}$ and selection of a future outer-time orientation are distinct representation-selection operations. They do not contribute to the local interruption energy and are not included in the term “interruption.”

Warning 11.3 (No bare interruption gap). A nonzero interruption defect is not yet a physical mass. Near-trivial holonomy gives arbitrarily small positive fiber eigenvalues, while the exact zero-interruption, trivial-holonomy stratum contains ordinary lattice Yang–Mills configurations. Therefore neither $\Phi_{\tau,n} > 0$ nor the first inner Fourier frequency yields a regulator-uniform Yang–Mills mass gap.

11.3 Projected inertia as residual symmetry-restoration cost

Near a coherent background, separate local variations into visible outer-record variables x and hidden inner-completion variables y . If the quadratic Hessian has block form

$$\mathcal{K} = \begin{pmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{pmatrix}, \quad K_{yy} > 0,$$

then stationary relaxation of y leaves the visible response

$$K_{\text{vis}} = K_{xx} - K_{xy} K_{yy}^{-1} K_{yx},$$

as proved abstractly in [theorem D.1](#). This quantity is the curvature of the effective phase after hidden inner-time channels have attempted to restore coherence. It is the controlled mathematical content of the claim that inertia is resistance to retuning or interruption of the inner-time flow.

The gauge-theory analogue is obtained after replacing the quadratic Hessian by the vacuum-compatible gap operator and allowing the high-interruption sector to relax. Define the punctured response operator

$$\mathfrak{M}_{\varepsilon,n}(\delta) := P_{\varepsilon,n}^{\circ} F_{\varepsilon,n}^{\text{vac}}(\delta) P_{\varepsilon,n}^{\circ}. \quad (72)$$

The notation $\mathfrak{M}$ records a *candidate mass stiffness*; it is not declared to be a mass until a regulator-uniform lower bound and a Yang–Mills continuum identification have been proved.

Proposition 11.4 (Interruption-response criterion for a finite-regulator gap). *Assume $QG_{n,\delta}Q > 0$. If*

$$\mathfrak{M}_{\varepsilon,n}(\delta) \geq c_n P_{\varepsilon,n}^\circ \quad (c_n > 0), \quad (73)$$

then the only zero vector of the full Feshbach operator in the low window is the transported vacuum line. If the corresponding high block is positive, $H_n \geq \delta$ on $\mathcal{N}_n \ominus \mathbb{C}\Omega_n$. A physical mass gap follows along a regulator family only if c_n has the required $a_{t,n}$ -scaled uniform lower bound and the continuum/identification theorems hold.

Proof. By theorem 12.3, the Feshbach operator annihilates the transported vacuum vector and reduces its orthogonal complement. Bound (73) makes the punctured low block strictly positive. Positivity of the high block and the Schur–Feshbach criterion then give $G_{n,\delta} \geq 0$; theorem 12.1 converts this to the stated Hamiltonian bound. The final sentence records the physical scaling and continuum requirements. $\square$

Remark 11.5 (The exact mass–geometry bridge). Within this manuscript, the phrase “mass from interrupted inner time” has one precise content:

$$\begin{array}{c} \text{inner-flow defect} \longrightarrow \text{hidden-mode relaxation} \\ \Downarrow \\ \text{positive punctured Schur–Feshbach response} \longrightarrow \text{outer-time spectral gap.} \end{array}$$

The first two arrows are exact constructions. The response becomes a theorem only after the uniform estimate in theorem 12.11; the final gap requires the continuum and Yang–Mills identification package. This formulation connects mass, temporal coherence, and spacetime readout without replacing an estimate by an interpretation.

12 Vacuum-compatible Feshbach reduction

Fix n and work on $\mathcal{N}_n$. Let

$$P_{\Omega,n} = |\Omega_n\rangle\langle\Omega_n|. \quad (74)$$

For $\delta > 0$, put

$$r_{n,\delta} = e^{-a_{t,n}\delta} \quad (75)$$

and define the required vacuum-compatible operator

$$G_{n,\delta} = r_{n,\delta}I - T_n + (1 - r_{n,\delta})P_{\Omega,n}. \quad (76)$$

Proposition 12.1 (Exact transfer formulation of a gap). *The operator $G_{n,\delta}$ is bounded and self-adjoint,*

$$G_{n,\delta}\Omega_n = 0, \quad G_{n,\delta}|_{\Omega_n^\perp} = r_{n,\delta}I - T_n.$$

Consequently

$$G_{n,\delta} \geq 0 \iff H_n \geq \delta I \text{ on } \mathcal{N}_n \ominus \mathbb{C}\Omega_n. \quad (77)$$

Proof. The vacuum identity follows from $T_n\Omega_n = \Omega_n$. Self-adjointness of T_n makes $\mathbb{C}\Omega_n \oplus \Omega_n^\perp$ reducing, and the rank-one correction vanishes on the second summand. On a spectral value $t \in (0, 1]$ of T_n , the corresponding energy is $-a_{t,n}^{-1} \log t$. Since the exponential is decreasing, $t \leq e^{-a_{t,n}\delta}$ is equivalent to energy at least δ . $\square$

Fix $\varepsilon > 0$ and abbreviate

$$P = P_{\varepsilon,n}, \quad Q = I - P, \quad G = G_{n,\delta}, \quad r = r_{n,\delta}.$$

Whenever

$$D := QGQ \geq dQ \quad (d > 0), \quad (78)$$

define the vacuum-compatible Feshbach operator

$$F_{\varepsilon,n}^{\text{vac}}(\delta) = PGP - PGQD^{-1}QGP \quad \text{on } P\mathcal{N}_n. \quad (79)$$

Theorem 12.2 (Bounded Schur–Feshbach criterion). *Under equation (78),*

$$G \geq 0 \iff F_{\varepsilon,n}^{\text{vac}}(\delta) \geq 0. \quad (80)$$

Together with theorem 12.1, either condition is equivalent to a gap at least δ for H_n on the neutral vacuum-orthogonal sector.

Proof. Relative to $\mathcal{N}_n = P\mathcal{N}_n \oplus Q\mathcal{N}_n$, write

$$G = \begin{pmatrix} A & C \\ C^* & D \end{pmatrix}.$$

For $x \in P\mathcal{N}_n$ and $y \in Q\mathcal{N}_n$, direct completion of the square gives

$$\begin{aligned} \langle x + y, G(x + y) \rangle &= \langle x, (A - CD^{-1}C^*)x \rangle \\ &\quad + \langle y + D^{-1}C^*x, D(y + D^{-1}C^*x) \rangle. \end{aligned} \quad (81)$$

The first operator is equation (79); the second term is nonnegative and can be made zero for each x . This proves the equivalence. $\square$

12.1 The transported vacuum line

Set

$$\omega = P\Omega_n, \quad \zeta = Q\Omega_n, \quad p = \|\omega\|^2, \quad q = \|\zeta\|^2 = 1 - p. \quad (82)$$

Proposition 12.3 (Vacuum transport through the high block). *Under equation (78), $\omega \neq 0$ and*

$$\zeta = -D^{-1}QG\omega. \quad (83)$$

Moreover

$$F_{\varepsilon,n}^{\text{vac}}(\delta)\omega = 0. \quad (84)$$

If

$$P_{\omega,n} = \frac{|\omega\rangle\langle\omega|}{\|\omega\|^2}, \quad P_{\varepsilon,n}^\circ = P - P_{\omega,n}, \quad (85)$$

then $P_{\varepsilon,n}^\circ$ reduces the self-adjoint Feshbach operator.

Proof. If $\omega = 0$, then $\Omega_n = \zeta \in Q\mathcal{N}_n$ and $D\Omega_n = QG\Omega_n = 0$, contradicting equation (78). The Q -component of $G(\omega + \zeta) = G\Omega_n = 0$ gives equation (83). The P -component gives

$$PGP\omega + PGQ\zeta = 0.$$

Substitution of equation (83) is exactly equation (84). Since the Feshbach operator is self-adjoint, a vector in its kernel has zero matrix element with its orthogonal complement; hence $\mathbb{C}\omega$ and $P_{\varepsilon,n}^\circ\mathcal{N}_n$ reduce it. $\square$

12.2 The exact obstruction

We now prove theorem 1.1 and apply it to the frozen regulator.

Proof of theorem 1.1. Write $\alpha = 1 - r > 0$, $p = \|\omega\|^2 > 0$, and $\zeta = Q\Omega \neq 0$. The Q -component of $G\Omega = 0$ is

$$QG\omega + D\zeta = 0. \quad (86)$$

On the other hand,

$$QG\omega = -QT\omega + \alpha Q|\Omega\rangle\langle\Omega|\omega = -QT\omega + \alpha p\zeta.$$

Combining this identity with equation (86) gives

$$QT\omega = (D + \alpha p I)\zeta. \quad (87)$$

The Schur complement annihilates ω by the same calculation as in theorem 12.3. Therefore

$$\langle\omega, PGP\omega\rangle = \langle QG\omega, D^{-1}QG\omega\rangle = \langle\zeta, D\zeta\rangle > 0. \quad (88)$$

Using equation (87),

$$\begin{aligned} \langle\omega, PTQD^{-1}QTP\omega\rangle &= \langle(D + \alpha p I)\zeta, D^{-1}(D + \alpha p I)\zeta\rangle \\ &= \langle\zeta, D\zeta\rangle + 2\alpha p\|\zeta\|^2 + \alpha^2 p^2\langle\zeta, D^{-1}\zeta\rangle. \end{aligned} \quad (89)$$

Both additional terms are strictly positive because $\alpha > 0$, $p > 0$, $\zeta \neq 0$, and $D^{-1} > 0$. Comparing equation (89) with equation (88) proves equation (1). The quadratic-form inequality equation (2) evaluated on ω would give a contradiction for every $\eta \leq 1$. $\square$

Corollary 12.4 (Failure of requested item (R3) in the frozen family). *Fix $\varepsilon_0 > 0$, $\delta > 0$, and any sufficiently large n . Let $P = P_{\varepsilon_0, n}$ and $Q = I - P$. If the high block $D = QG_{n, \delta}Q$ is strictly positive, then no constant $\eta \leq 1$ satisfies*

$$PT_n QD^{-1}QT_n P \leq \eta PG_{n, \delta}P \quad (90)$$

on $P\mathcal{N}_n$.

Proof. By theorem 10.10, both $P\Omega_n$ and $Q\Omega_n$ are nonzero for all sufficiently large n . Apply theorem 1.1. $\square$

Remark 12.5 (Exceptional choices are not the requested split). If $\varepsilon_0 \geq \|K_n\|$, then $Q = 0$ and there is no high block to invert; the proposed reduction collapses to the full global gap problem. If $\varepsilon_0 = 0$, the low multiplication projection is zero because the exact zero set has Haar measure zero by theorem 10.7. The request specifies $\varepsilon_0 > 0$ independent of the regulator, and $\|K_n\| \rightarrow \infty$, so neither exceptional case avoids theorem 12.4 along the frozen family.

Proposition 12.6 (A necessary vacuum-tail condition for direct high contraction). *Let $T = T^* \geq 0$, $T\Omega = \Omega$, and $P + Q = I$. If*

$$QTQ \leq \rho Q \quad (0 \leq \rho < 1), \quad (91)$$

then, with $p = \|P\Omega\|^2$,

$$p \geq \frac{1 - \rho}{2 - \rho}. \quad (92)$$

Proof. Let $\omega = P\Omega$ and $\zeta = Q\Omega$. From $T\Omega = \Omega$,

$$\langle \omega, T\zeta \rangle = p - \langle \omega, T\omega \rangle.$$

Expanding $1 = \langle \Omega, T\Omega \rangle$ and using self-adjointness gives

$$\langle \zeta, T\zeta \rangle = 1 - 2p + \langle \omega, T\omega \rangle \geq 1 - 2p,$$

because $T \geq 0$. The assumed high contraction gives

$$1 - 2p \leq \rho \|\zeta\|^2 = \rho(1 - p).$$

Rearrangement yields equation (92). $\square$

For $\rho = e^{-a_{t,n}m_Q}$, equation (92) requires the low window to contain at least a definite $O(a_{t,n})$ portion of the vacuum norm. This is a necessary condition only; it does not prove the high estimate.

12.3 The corrected punctured implication

The obstruction comes solely from applying the T -mixing inequality to the transported vacuum line. On the punctured low space the rank-one correction disappears.

Lemma 12.7 (Equality of the punctured T - and G -mixing forms). *Let $P^\circ = P_{\varepsilon,n}^\circ$. Then*

$$QGP^\circ = -QT_nP^\circ, \quad P^\circ GQ = -P^\circ T_nQ, \quad (93)$$

and

$$P^\circ GP^\circ = rP^\circ - P^\circ T_nP^\circ. \quad (94)$$

Proof. For $x \in \text{Ran } P^\circ$,

$$\langle \Omega_n, x \rangle = \langle P\Omega_n, x \rangle = \langle \omega, x \rangle = 0.$$

Thus $P_{\Omega,n}x = 0$. Since $Qx = 0$, the rI term has no off-diagonal block, proving equation (93). Compression to P° gives equation (94). $\square$

Proposition 12.8 (Corrected conditional two-window theorem). *Assume constants $m_0 > \delta > 0$, $m_Q > \delta$, and $0 \leq \eta < 1$ satisfy, at a fixed regulator,*

$$P^\circ T_n P^\circ \leq e^{-a_{t,n}m_0} P^\circ, \quad (95)$$

$$QT_nQ \leq e^{-a_{t,n}m_Q} Q, \quad (96)$$

$$P^\circ PT_nQ [QGQ]^{-1} QT_n P P^\circ \leq \eta P^\circ GP^\circ. \quad (97)$$

Then QGQ is invertible and

$$F_{\varepsilon,n}^{\text{vac}}(\delta) \geq (1 - \eta) \left(e^{-a_{t,n}\delta} - e^{-a_{t,n}m_0} \right) (P - P_{\omega,n}). \quad (98)$$

If the same constants hold for a family with $0 < a_{t,n} \leq a_t^{\max}$, then

$$F_{\varepsilon,n}^{\text{vac}}(\delta) \geq c a_{t,n} (P - P_{\omega,n}), \quad (99)$$

where the explicit regulator-independent constant is

$$c = (1 - \eta)(m_0 - \delta)e^{-a_t^{\max}m_0} > 0. \quad (100)$$

Proof. By equation (96) and positivity of the rank-one vacuum correction,

$$\begin{aligned} QGQ &= rQ - QT_nQ + (1-r)|\zeta\rangle\langle\zeta| \\ &\geq \left(e^{-a_{t,n}\delta} - e^{-a_{t,n}m_Q}\right)Q > 0. \end{aligned}$$

Thus the inverse in equation (97) exists. By theorem 12.7, the Feshbach compression is

$$\begin{aligned} P^\circ F P^\circ &= P^\circ G P^\circ - P^\circ P T_n Q (QGQ)^{-1} Q T_n P P^\circ \\ &\geq (1-\eta) P^\circ G P^\circ. \end{aligned}$$

The low contraction and equation (94) give

$$P^\circ G P^\circ \geq \left(e^{-a_{t,n}\delta} - e^{-a_{t,n}m_0}\right) P^\circ.$$

The Feshbach operator annihilates ω and has no cross term with P° by theorem 12.3, proving equation (98). Finally,

$$\begin{aligned} e^{-a_{t,n}\delta} - e^{-a_{t,n}m_0} &= a_{t,n} \int_\delta^{m_0} e^{-a_{t,n}s} ds \\ &\geq a_{t,n}(m_0 - \delta)e^{-a_t^{\max}m_0}, \end{aligned}$$

which gives equations (99) and (100). $\square$

12.4 Symmetry covariance of the punctured Feshbach map

Proposition 12.9 (Symmetry covariance). *Let $U_n : \mathcal{S}_n \rightarrow U(\mathcal{N}_n)$ fix Ω_n and commute with T_n and K_n . Then U_n commutes with*

$$P_{\varepsilon,n}, \quad Q_{\varepsilon,n}, \quad G_{n,\delta}, \quad P_{\omega,n}, \quad P_{\varepsilon,n}^\circ, \quad F_{\varepsilon,n}^{\text{vac}}(\delta), \quad \mathfrak{M}_{\varepsilon,n}(\delta).$$

Consequently every invariant finite-rank dangerous cluster of $\mathfrak{M}_{\varepsilon,n}(\delta)$ admits the real-, complex-, and quaternionic-type block decomposition of theorem 4.1.

Proof. Commutation with $P_{\varepsilon,n}$ and $Q_{\varepsilon,n}$ follows from the functional calculus of K_n . The vacuum projection commutes because the vacuum is fixed. Hence $G_{n,\delta}$ and its compressed high block commute, as does the inverse on its range. The Feshbach expression is therefore equivariant. Since $P\Omega_n$ is fixed, so are $P_{\omega,n}$ and the punctured projection. Apply theorem 4.1 to any finite-rank invariant cluster. $\square$

Program Requirement 12.10 (Symmetry-block interruption rigidity). Choose a regulator-stable $\mathcal{S}_n$ -invariant dangerous cluster in the punctured low window. In the decomposition

$$\Pi_n P_{\varepsilon_0,n}^\circ \mathcal{N}_{n,\mathbb{R}} \cong \bigoplus_{\sigma} V_{\sigma} \otimes_{\mathbb{D}_{\sigma}} M_{\sigma,n},$$

prove, with constants independent of n , that every nonvacuum multiplicity operator obeys

$$\mathfrak{M}_{\sigma,n}(\delta) \geq c a_{t,n} I_{M_{\sigma,n}}, \quad c > 0,$$

and that the only trivial-isotypic zero mode is the transported vacuum. The estimate must include all center/flux sectors and control the complement of the cluster.

This target is the symmetry-resolved version of the mass-from-interruption claim. The compact symmetry does not produce the constant c ; it reduces the proof to multiplicity spaces on which the nonabelian dynamics must be estimated.

Program Requirement 12.11 (Regulator-uniform punctured estimates). Fix $\varepsilon_0, m_0, m_Q, \delta, \eta$ with $m_0, m_Q > \delta > 0$ and $\eta < 1$. Prove equations (95) to (97) for every sufficiently large n , with $P = P_{\varepsilon_0, n}$.

Proof Audit 12.12 (Status of the repaired target). No proof of theorem 12.11 is supplied by the finite-regulator construction, the holonomy spectrum, the classical zero-set classification, the DTT dimension module, or the response-Schur analogy. By theorem 10.4, the low window contains neighborhoods of the ordinary lattice Yang–Mills stratum. Proving equation (95) there is a genuine nonabelian infrared estimate, not an auxiliary Fourier estimate. The target therefore remains unproved.

Part II

Completion-band redesign

13 From the hard-window obstruction to a spectral-band regulator

The hard multiplication window of section 12 is useful diagnostically: it identifies configurations with small mixed interruption energy and reveals the transported-vacuum obstruction. It is not a satisfactory local quantum decomposition. Two facts force a redesign.

First, theorems 1.1 and 12.4 show that the unpunctured relative-mixing estimate is algebraically impossible whenever a nonconstant multiplication window splits the positive vacuum. Second, theorem 10.7 shows that a nonnegative plaquette multiplication operator may have spectral infimum zero but no nonzero L^2 ground vector. Thus a frustration-free local-projector argument cannot begin from the exact zero set of the multiplication potential.

The replacement is to form a local Schrödinger operator in the auxiliary completion variables. Its kinetic term turns a classical well into a genuine quantum band. A separate holonomy-closure term removes continuous flat auxiliary moduli. The band projection, rather than the hard multiplication projection, becomes the object patched and eliminated in the low-energy proof.

13.1 Gauge-invariant holonomy closure

For a slice configuration, recall the auxiliary holonomy

$$H_x = V(x, 0)V(x, 1) \cdots V(x, N_{\tau, n} - 1).$$

Define

$$\mathcal{H}_n(V) = \sum_{x \in \Sigma_n} \ell_\rho(H_x). \quad (101)$$

Proposition 13.1 (Symmetry of the closure functional). *The function $\mathcal{H}_n$ is continuous, nonnegative, gauge invariant, and invariant under cyclic inner-time translation. It vanishes exactly when $H_x = e$ for every spatial site.*

Proof. Under a slice gauge transformation, $H_x \mapsto g(x, 0)H_x g(x, 0)^{-1}$, so conjugation invariance of ℓ_ρ gives gauge invariance. Changing the base point of the ordered product around the auxiliary cycle conjugates the holonomy by the transported link, so cyclic translation also preserves $\ell_\rho(H_x)$. Nonnegativity, continuity, and the characterization of the zero set follow from theorem 7.1. $\square$

Use the frozen sequence $\kappa_{H,n}$ of equation (21). Define the exact completion-band potential

$$\mathcal{V}_n^{\text{cb}} = \mathcal{V}_n + \kappa_{H,n} \mathcal{H}_n. \quad (102)$$

On the gauge-invariant Sobolev form domain

$$\mathcal{Q}_n^{\text{sl}} = H^1(\mathcal{X}_n, \mu_n) \cap \mathcal{H}_n^{\text{sl}},$$

define

$$q_{n,0}^{\text{cb}}[\Psi] = \|\mathcal{L}_n^{1/2} \Psi\|^2 + \langle \Psi, \mathcal{V}_n^{\text{cb}} \Psi \rangle. \quad (103)$$

Let $\mathbb{H}_{n,0}^{\text{cb}}$ be the Friedrichs operator associated with this form, let

$$E_{0,n}^{\text{cb}} = \inf \text{Spec } \mathbb{H}_{n,0}^{\text{cb}}, \quad H_n^{\text{cb}} = \mathbb{H}_{n,0}^{\text{cb}} - E_{0,n}^{\text{cb}} I, \quad T_n^{\text{cb}} = e^{-a_{t,n} H_n^{\text{cb}}}. \quad (104)$$

Set $\mathcal{H}_n^{\text{cb}} := \mathcal{H}_n^{\text{sl}}$.

Proposition 13.2 (Exact finite-regulator completion Hamiltonian). *For every fixed regulator, $\mathbb{H}_{n,0}^{\text{cb}}$ is self-adjoint, bounded below, and has compact resolvent. Its lowest eigenvalue is simple and has a smooth strictly positive normalized eigenfunction Ω_n^{cb} . The shifted Hamiltonian H_n^{cb} is nonnegative and self-adjoint, and*

$$T_n^{\text{cb}} = e^{-a_{t,n} H_n^{\text{cb}}}$$

is self-adjoint, positive, injective, trace class, positivity improving, and satisfies $T_n^{\text{cb}} \Omega_n^{\text{cb}} = \Omega_n^{\text{cb}}$. Its ground-state transform gives a reversible two-sided vacuum Markov process and a finite-regulator OS reconstruction.

Proof. The first term in (103) is the closed elliptic Sobolev form $\|\mathcal{L}_n^{1/2} \Psi\|^2$, and the second is a bounded real multiplication-form perturbation on the compact configuration manifold. The representation theorem gives a self-adjoint semibounded operator with compact resolvent. The Trotter product formula, or equivalently the Feynman–Kac formula for a bounded potential on a compact manifold, shows that $e^{-t \mathbb{H}_{n,0}^{\text{cb}}}$ is positivity improving for every $t > 0$. Jentzsch–Perron theory therefore makes the ground eigenvalue simple with a smooth strictly positive eigenfunction. Subtracting $E_{0,n}^{\text{cb}}$ gives a nonnegative operator. Functional calculus gives injectivity and trace class of its positive-time semigroup. The ground-state transform and the proof of theorem 9.2 then produce the stationary reversible vacuum process and OS representation. $\square$

Define the completion-band vacuum probability measure and ground-state transform by

$$d\nu_n^{\text{cb}} = |\Omega_n^{\text{cb}}|^2 d\mu_n, \quad U_{\Omega,n}^{\text{cb}} f = \Omega_n^{\text{cb}} f, \quad \hat{T}_n^{\text{cb}} = (U_{\Omega,n}^{\text{cb}})^{-1} T_n^{\text{cb}} U_{\Omega,n}^{\text{cb}}. \quad (105)$$

Corollary 13.3 (Exact completion-band vacuum Markov representation). *The map*

$$U_{\Omega,n}^{\text{cb}} : L^2(\mathcal{X}_n, \nu_n^{\text{cb}})^{\mathcal{G}_n} \longrightarrow \mathcal{H}_n^{\text{cb}}$$

is unitary. The operator $\hat{T}_n^{\text{cb}}$ is a self-adjoint positivity-improving Markov contraction,

$$\hat{T}_n^{\text{cb}} \mathbf{1} = \mathbf{1},$$

and its reversible two-sided stationary process reconstructs the same vacuum Hilbert space and Hamiltonian H_n^{cb} .

Proof. Unitarity follows from $d\nu_n^{\text{cb}} = |\Omega_n^{\text{cb}}|^2 d\mu_n$. The eigenvalue equation $T_n^{\text{cb}}\Omega_n^{\text{cb}} = \Omega_n^{\text{cb}}$ gives the Markov identity, while unitary conjugacy preserves self-adjointness, positivity improvement, and the spectrum. The stationary-process and Osterwalder–Schrader conclusion is the ground-state-transform argument of [theorem 9.2](#) with T_n, Ω_n replaced by $T_n^{\text{cb}}, \Omega_n^{\text{cb}}$. $\square$

Proposition 13.4 (Strang kernels approximate the exact semigroup at fixed regulator). *For fixed n and $t > 0$,*

$$\left(e^{-\frac{t}{2m}\mathcal{V}_n^{\text{cb}}} e^{-\frac{t}{m}\mathcal{L}_n} e^{-\frac{t}{2m}\mathcal{V}_n^{\text{cb}}} \right)^m \xrightarrow[m \rightarrow \infty]{s} e^{-t\mathbb{H}_{n,0}^{\text{cb}}}. \quad (106)$$

Every approximant has a strictly positive gauge-averaged heat kernel. No regulator-uniform rate in n is inferred from this fixed- n product formula.

Proof. The potential is bounded and the kinetic form is closed. The symmetric Trotter–Kato product formula gives strong convergence. Positivity of each factor and strict positivity of the heat kernel give the kernel statement. $\square$

Proof Audit 13.5 (Universality and slow–fast burdens of the exact regulator). The exact Hamiltonian removes the earlier ambiguity between the logarithm of a one-step Strang operator and a separately introduced local Schrödinger operator. It does not by itself identify the local fiber Hamiltonians with components of the global form. That identification is the slow–fast disintegration target [theorem 14.6](#). Moreover, the closure potential is an auxiliary regulator device, not part of the target four-dimensional Yang–Mills action. A complete proof must show that it opens a diverging transverse band while all induced relevant and marginal four-dimensional couplings are forbidden, renormalized into the ordinary Yang–Mills parameters, or vanish. This is the compatibility problem in [theorems 19.4, 19.5 and 31.1](#).

13.2 Design criteria for the new low-energy decomposition

The completion-band construction must satisfy five requirements.

- (B1) *Spectral reality:* local low spaces are ranges of spectral projections of self-adjoint completion Hamiltonians, not supports of measure-zero classical zero sets.
- (B2) *Gauge covariance:* band projections commute with local gauge constraints and with the exact compact regulator symmetries.
- (B3) *Transverse character:* the band removes only relative auxiliary modes; the physical τ -constant gauge field remains as a base variable.
- (B4) *Uniformity:* the transverse gap is bounded below in physical units uniformly in auxiliary resolution and physical refinement.
- (B5) *Universality:* eliminating the band produces pure four-dimensional Yang–Mills plus controlled irrelevant gauge-invariant corrections.

These criteria define the specifically DTT portion of the program.

14 Local completion Hamiltonians

14.1 Orbit types and stratified gauge quotients

The local gauge action is not free on all configurations. Reducible configurations have nontrivial stabilizers, and the quotient is generally a stratified space rather than a smooth manifold. The slow–fast construction must therefore be formulated orbit type by orbit type.

Definition 14.1 (Orbit-type stratum). Let a compact Lie group K act smoothly on a manifold X . For a closed subgroup $H \leq K$, let

$$X_{(H)} = \{x \in X : K_x \text{ is conjugate to } H\}.$$

The image $X_{(H)}/K$ is the orbit-type stratum of type (H) .

Proposition 14.2 (Fixed-regulator orbit-type decomposition). *For every fixed finite block B , regulator n , and boundary/flux sector σ , the compact local configuration manifold has only finitely many orbit types under the residual compact gauge group. Each orbit-type subset is a smooth invariant submanifold, and its quotient is a smooth stratum. A neighborhood of each orbit admits a compact-group slice model.*

Proof. This is the compact transformation-group slice theorem and finiteness of orbit types for a smooth action of a compact Lie group on a compact manifold; see [51]. Applied to the residual local gauge action, it yields a finite orbit-type decomposition at each fixed regulator and the corresponding local slice models. $\square$

Program Requirement 14.3 (Uniform stratified slow-fast atlas). Construct the slow-fast atlas simultaneously on all orbit-type strata and prove constants uniform under refinement. On overlaps, the conditional fast forms, Jacobians, band projections, and cross forms must transform covariantly. Near lower-dimensional singular strata, prove either uniform extension of the estimates or an energy/measure bound showing that the singular neighborhood contributes a controlled error. No argument may assume that the full gauge quotient is globally smooth.

14.2 Blocks, base variables, and completion fibers

Let $B \subset \Sigma_n$ be a finite connected spatial block. Fix a boundary condition and a topological/-center sector σ . In a tubular neighborhood of the coherent stratum, choose an auxiliary axial gauge and a local gauge slice. The remaining coordinates are separated into

$$u \in \mathcal{U}_{B,n}^{(\sigma)}, \quad y \in \mathcal{Y}_{B,n}^{(\sigma)}(u),$$

where u records the τ -constant physical links and y records relative inner-time deformations, auxiliary links, and holonomy variables normal to the coherent set. The use of a local slice is harmless at fixed block size; uniform control over overlapping slices is part of the nonlinear band target.

Let $\mu_{B,n,u}^{\text{aux}}$ be the induced smooth positive measure on the compact completion fiber and set

$$\mathcal{H}_{B,n}^{\text{aux}}(u) = L^2\left(\mathcal{Y}_{B,n}^{(\sigma)}(u), \mu_{B,n,u}^{\text{aux}}\right).$$

Let $L_{\tau,B,n}^{\text{rel}}(u)$ be the nonnegative Laplace–Beltrami operator in all relative completion directions. Tangent directions corresponding to physical zero modes and gauge orbits are excluded from this operator by construction.

Definition 14.4 (Local completion form and Hamiltonian). For $\psi \in H^1(\mathcal{Y}_{B,n}^{(\sigma)}(u))$, define

$$c_{B,n,u}^{(\sigma)}[\psi] = c_{\tau,n}^{\text{kin}} \|L_{\tau,B,n}^{\text{rel}}(u)^{1/2}\psi\|^2 \quad (107)$$

$$+ \int_{\mathcal{Y}_{B,n}^{(\sigma)}(u)} \left[g_n^{-2} \mathcal{K}_{B,n}(u, y) + \kappa_{H,n} \mathcal{H}_{B,n}(y) \right] |\psi(y)|^2 d\mu_{B,n,u}^{\text{aux}}(y), \quad (108)$$

where $c_{\tau,n}^{\text{kin}}$ is the exact relative auxiliary kinetic coefficient fixed in equation (21). The Friedrichs operator associated with this form is denoted

$$\mathcal{C}_{B,n}^{(\sigma)}(u). \quad (109)$$

Proposition 14.5 (Fixed-regulator operator theory). *For every fixed B, n, u, σ , the form (107) is densely defined, closed, and nonnegative. The operator $\mathcal{C}_{B,n}^{(\sigma)}(u)$ is self-adjoint, nonnegative, and has compact resolvent. Hence its spectrum is a sequence of eigenvalues of finite multiplicity tending to infinity.*

Proof. The fiber is compact and the induced metric and density are smooth and positive. The quadratic form $\|L_{\tau,B,n}^{\text{rel}}(u)^{1/2}\psi\|^2$ is the closed Sobolev form of the nonnegative elliptic operator. The potential in square brackets is continuous and bounded below by zero, hence is a bounded form perturbation. The sum is closed and nonnegative. Compactness of the embedding $H^1 \hookrightarrow L^2$ on a compact finite-dimensional manifold gives compact resolvent. $\square$

Program Requirement 14.6 (Gauge-covariant slow-fast form disintegration). Assume the stratified atlas of [theorem 14.3](#). Construct, on a finite equivariant atlas of each declared block and sector, unitary chart maps under which the exact global form $q_{n,0}^{\text{cb}}$ admits a slow-fast decomposition

$$q_{n,0}^{\text{cb}}[\Psi] = q_{B,n}^{\text{phys}}[\Psi] + \int_{\mathcal{U}_{B,n}^{(\sigma)}} c_{B,n,u}^{(\sigma)}[\Psi(u, \cdot)] d\varpi_{B,n}^{(\sigma)}(u) + 2 \operatorname{Re} b_{B,n}[\Psi] + r_{B,n}[\Psi]. \quad (110)$$

The construction must satisfy all of the following.

- (SF1) The fast form in [equation \(110\)](#) is exactly the conditional auxiliary part of $q_{n,0}^{\text{cb}}$, with coefficient $c_{\tau,n}^{\text{kin}}$, not an independently postulated model.
- (SF2) Gauge-orbit directions and physical τ -constant base directions are separated from the fast normal bundle, and the chart transition maps preserve the Gauss constraint and the declared flux sector.
- (SF3) The cross form has a regulator-uniform relative estimate in the form norms,

$$|b_{B,n}[p, q]| \leq \eta_{B,n}(q_{B,n}^{\text{phys}}[p] + \Lambda_* \|p\|^2)^{1/2} (c_{B,n}^{(\sigma)}[q] + \Lambda_* \|q\|^2)^{1/2}, \quad (111)$$

with a quantitative sequence suitable for [theorem 18.5](#).

- (SF4) The remainder $r_{B,n}$ is relatively form bounded with a bound tending to zero after the fast gap is inserted.
- (SF5) The decomposition is compatible on chart overlaps and reconstructs the original exact form without double counting overlapping blocks.

Proof Audit 14.7 (Why disintegration is indispensable). Compact resolvent of $\mathcal{C}_{B,n}^{(\sigma)}(u)$ does not make it a spectral component of H_n^{cb} . The local band can influence the physical Hamiltonian only after [equation \(110\)](#) and the overlap identities have been proved. This target is therefore logically prior to the nonlinear band and global patching theorems.

14.3 Gauge and regulator symmetry

Let $\mathcal{S}_{B,n}^{(\sigma)}$ denote the compact group generated by the residual boundary-preserving gauge symmetry, block rotations, inner translations, and center transformations that preserve σ .

Proposition 14.8 (Equivariance of the completion Hamiltonian). *Assume the local gauge slice and fiber metric are chosen equivariantly. Then*

$$[\mathcal{C}_{B,n}^{(\sigma)}(u), U_{B,n}^{(\sigma)}(s)] = 0 \quad (s \in \mathcal{S}_{B,n}^{(\sigma)}).$$

Every isolated spectral-band projection of $\mathcal{C}_{B,n}^{(\sigma)}(u)$ is therefore equivariant and decomposes into real, complex, and quaternionic isotypic blocks as in [theorem 4.1](#).

Proof. The Laplace operator is constructed from an invariant metric, while $\mathcal{K}_{B,n}$ and $\mathcal{H}_{B,n}$ are invariant class functions of plaquette and holonomy data. The form is therefore invariant. Commutation of the associated operator and its Borel functional calculus follows from uniqueness in the representation theorem for closed forms. $\square$

14.4 Ground bands

Write the ordered eigenvalues of $\mathcal{C}_{B,n}^{(\sigma)}(u)$ as

$$E_{0,B,n}^{(\sigma)}(u) \leq E_{1,B,n}^{(\sigma)}(u) \leq \cdots$$

A completion ground band of rank $r_{B,n}^{(\sigma)}$ is an interval containing the first $r_{B,n}^{(\sigma)}$ eigenvalues and separated from the rest. Its spectral projection is denoted

$$\Pi_{B,n}^{(\sigma)}(u).$$

The band need not be one dimensional in nontrivial boundary or flux sectors. The correct requirement is that its rank be controlled and that every nonphysical degeneracy be classified by symmetry.

15 Coherent histories and the interruption normal form

15.1 The coherent completion set

Define the local coherent set before quotienting by

$$\tilde{\mathcal{Z}}_{B,n}^{(\sigma)} = \{(U, V) : W_{i\tau} = e \text{ on } B, H_x = e \text{ on } B\}. \quad (112)$$

After imposing the boundary sector and dividing by the local gauge group, denote the resulting set by $\mathcal{Z}_{B,n}^{(\sigma)}$. On the trivial-holonomy branch, [theorems 10.3](#) and [10.4](#) identify $\mathcal{Z}_{B,n}^{(\sigma)}$ with the ordinary four-dimensional gauge variables restricted to one outer-time slice.

The closure term is essential here. Without it, the zero-mixed set contains continuous covariantly constant holonomy strata. The closure potential reduces the target normal manifold to the ordinary physical zero modes and discrete topological data.

15.2 A proved linearized stiffness theorem

The next lemma is the exact finite-dimensional core of the completion-band idea.

Lemma 15.1 (Cyclic Poincaré estimate). *Let E be a finite-dimensional real or complex Hilbert space, let $N \geq 2$, $a_\tau > 0$, and $\beta_\tau = Na_\tau$. For $\xi = (\xi_0, \dots, \xi_{N-1}) \in E^N$, set*

$$\bar{\xi} = \frac{1}{N} \sum_{r=0}^{N-1} \xi_r.$$

Then

$$\sum_{r=0}^{N-1} \frac{\|\xi_{r+1} - \xi_r\|^2}{a_\tau^2} \geq \frac{16}{\beta_\tau^2} \sum_{r=0}^{N-1} \|\xi_r - \bar{\xi}\|^2, \quad (113)$$

with cyclic indices.

Proof. Discrete Fourier transformation diagonalizes the left side. On the mode k , the eigenvalue is

$$4a_\tau^{-2} \sin^2\left(\frac{\pi k}{N}\right).$$

The constant mode is the mean. For $1 \leq k \leq N-1$, symmetry reduces to $1 \leq k \leq N/2$, and $\sin x \geq 2x/\pi$ on $[0, \pi/2]$ gives

$$4a_\tau^{-2} \sin^2(\pi/N) \geq 16(Na_\tau)^{-2}.$$

Summing the nonzero Fourier modes proves the result. $\square$

For the exact regulator, the quadratic expansion of the mixed potential around a coherent trivial-holonomy history has the normalized coefficient

$$\alpha = \alpha_n^{\text{mix}} = \frac{a_n}{2g_n^2 N_{\tau,n}}, \quad \kappa = \gamma_{H,n}^{\text{lin}} = \frac{1}{2} \kappa_{H,n} N_{\tau,n}. \quad (114)$$

Indeed, the mixed plaquette expansion contributes $\frac{a_n}{2g_n^2 N_{\tau,n}} \sum_r \|\xi_{r+1} - \xi_r\|^2 / a_{\tau,n}^2$, while

$$\kappa_{H,n} \ell_\rho \left(\exp \sum_r \xi_r \right) = \frac{1}{2} \kappa_{H,n} N_{\tau,n}^2 \|\bar{\xi}\|^2 + O(\|\xi\|^3).$$

Theorem 15.2 (Linearized transverse completion stiffness). *Let $\alpha, \kappa > 0$. Define on E^N*

$$q_{\alpha,\kappa}(\xi) = \alpha \sum_{r=0}^{N-1} \frac{\|\xi_{r+1} - \xi_r\|^2}{a_\tau^2} + \kappa N \|\bar{\xi}\|^2. \quad (115)$$

Then

$$q_{\alpha,\kappa}(\xi) \geq \min \left\{ \frac{16\alpha}{\beta_\tau^2}, \kappa \right\} \sum_{r=0}^{N-1} \|\xi_r\|^2. \quad (116)$$

Proof. The orthogonal decomposition $\xi_r = \bar{\xi} + (\xi_r - \bar{\xi})$ gives

$$\sum_r \|\xi_r\|^2 = N \|\bar{\xi}\|^2 + \sum_r \|\xi_r - \bar{\xi}\|^2.$$

The difference term annihilates the mean and satisfies equation (113) on the zero-mean part. The closure term is exactly κ times the squared norm of the mean component. Taking the smaller of the two coefficients proves (116). $\square$

Corollary 15.3 (Diverging frozen tangent band). *Under the frozen scaling in (21), the tangent normal-form floor is*

$$\Gamma_n^{\text{lin}} = \min \left\{ \frac{8\Lambda_*}{g_n^2}, \frac{\kappa_* \Lambda_*}{2} N_{\tau,n}^{\sigma_H} \right\} \longrightarrow \infty. \quad (117)$$

Every zero-mean relative mode is bounded below by $8\Lambda_*/g_n^2$, while the coherent mean/holonomy mode is bounded below by $(\kappa_* \Lambda_*/2) N_{\tau,n}^{\sigma_H}$.

Proof. Apply theorem 15.2 with the coefficients in equation (114) and use theorem 7.4. $\square$

This result is local and quadratic. It does not by itself control large excursions, gauge-slice transitions, multiple nonlinear wells, or coupling to the physical base variables.

15.3 Nonlinear normal-form target

Program Requirement 15.4 (Uniform gauge-covariant Morse–Bott normal form). Assume theorem 14.6. For every fixed physical block size, every allowed boundary/flux sector σ , and every sufficiently fine regulator, construct finitely many equivariant gauge slices covering a tubular neighborhood of $\mathcal{Z}_{B,n}^{(\sigma)}$. In each slice, write fast normal coordinates ξ such that

$$g_n^{-2} \mathcal{K}_{B,n} + \kappa_{H,n} \mathcal{H}_{B,n} = \langle \xi, \mathcal{Q}_{B,n}^{(\sigma)}(u) \xi \rangle + R_{B,n}^{(\sigma)}(u, \xi), \quad (118)$$

with the following regulator-uniform properties.

(MB1) On the purely auxiliary normal bundle,

$$\mathcal{Q}_{B,n}^{(\sigma)}(u) \geq q_{\perp,n} I, \quad q_{\perp,n} \geq c_{\perp} \Gamma_n^{\text{lin}}, \quad q_{\perp,n} \rightarrow \infty. \quad (119)$$

(MB2) In a regulator-independent normal tube,

$$|R_{B,n}^{(\sigma)}(u, \xi)| \leq \theta \langle \xi, \mathcal{Q}_{B,n}^{(\sigma)}(u) \xi \rangle, \quad 0 \leq \theta < 1, \quad (120)$$

and the first two base derivatives obey corresponding relative bounds.

(MB3) Outside that tube,

$$g_n^{-2} \mathcal{K}_{B,n} + \kappa_{H,n} \mathcal{H}_{B,n} \geq q_{\text{far},n}, \quad q_{\text{far},n} \rightarrow \infty \quad (121)$$

for configurations that are not physical tangent directions.

(MB4) Gauge tangent directions are removed exactly. The tangent space of the coherent manifold consists precisely of physical τ -constant base directions and the declared discrete topological degeneracies.

(MB5) The gauge-slice metrics, densities, Faddeev–Popov Jacobians, and transition maps have uniform C^2 bounds after physical rescaling, and no additional remote low well is omitted.

Proof Audit 15.5 (What the linear theorem does and does not prove). [Theorem 15.2](#) proves the desired stiffness in the tangent model at trivial holonomy. Extending it to [theorem 15.4](#) requires uniform exponential-coordinate control on a growing lattice, quantitative treatment of all holonomy strata, and a gauge-covariant description of the normal bundle. None of those follows from Fourier diagonalization alone.

15.4 An abstract perturbative normal-form lemma

Lemma 15.6 (Stability of a coercive quadratic normal form). *Let $q_0(\xi) \geq \gamma \|\xi\|^2$ and suppose $|R(\xi)| \leq \theta q_0(\xi)$ on a neighborhood, with $0 \leq \theta < 1$. Then*

$$q_0(\xi) + R(\xi) \geq (1 - \theta) \gamma \|\xi\|^2.$$

Proof. The assumed relative bound gives $R \geq -\theta q_0$. Hence $q_0 + R \geq (1 - \theta) q_0 \geq (1 - \theta) \gamma \|\xi\|^2$. $\square$

16 Local transverse spectral bands

16.1 Fixed-regulator band projections

For fixed B, n, u, σ , compact resolvent ensures a discrete spectrum. Choose a contour Γ enclosing a finite group of low eigenvalues but no other spectrum, and define

$$\Pi_{B,n}^{(\sigma)}(u) = \frac{1}{2\pi i} \int_{\Gamma} (z - \mathcal{C}_{B,n}^{(\sigma)}(u))^{-1} dz. \quad (122)$$

Proposition 16.1 (Norm-resolvent stability of a band). *Let $u \mapsto \mathcal{C}(u)$ be a family of self-adjoint compact-resolvent operators such that the resolvents are norm-continuous at one nonreal spectral parameter. Suppose a fixed contour Γ lies in the resolvent set for all u in a connected parameter set. Then the Riesz projection $\Pi(u)$ is norm-continuous and has constant finite rank.*

Proof. The resolvent identity transfers norm-continuity from one nonreal point to every point on the compact contour. Integration in (122) gives norm-continuity of $\Pi(u)$. Projections at norm distance less than one have the same rank; connectedness and a finite open-cover argument make the rank constant. $\square$

Proposition 16.2 (Symmetry decomposition of a band). *If a compact group S commutes with $\mathcal{C}(u)$, then it commutes with $\Pi(u)$. On the realification of its finite-dimensional range,*

$$\text{Ran } \Pi(u) \simeq \bigoplus_{\sigma} V_{\sigma} \otimes_{\mathbb{D}_{\sigma}} M_{\sigma}(u), \quad \mathbb{D}_{\sigma} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}.$$

Every equivariant effective operator is tested on the multiplicity spaces.

Proof. The resolvent commutes with S , hence so does the contour integral. The rest is theorem 4.1. $\square$

16.2 IMS localization

The nonlinear band proof requires separate estimates near and far from the coherent manifold.

Lemma 16.3 (Two-region IMS lower bound). *Let q be a closed nonnegative form and let χ_0, χ_1 be bounded form multipliers satisfying $\chi_0^2 + \chi_1^2 = 1$. Suppose*

$$q[f] = q[\chi_0 f] + q[\chi_1 f] - \mathcal{E}_{\text{IMS}}[f], \quad 0 \leq \mathcal{E}_{\text{IMS}}[f] \leq e_{\text{IMS}} \|f\|^2. \quad (123)$$

If

$$q[\chi_j f] \geq m_j \|\chi_j f\|^2 \quad (j = 0, 1),$$

then

$$q[f] \geq (\min\{m_0, m_1\} - e_{\text{IMS}}) \|f\|^2. \quad (124)$$

Proof. Insert the two local lower bounds into (123) and use

$$\|\chi_0 f\|^2 + \|\chi_1 f\|^2 = \|f\|^2.$$

$\square$

For a Laplace-type form, the IMS error is an explicit integral of $|\nabla \chi_0|^2 + |\nabla \chi_1|^2$. A wider transition annulus reduces this error, but the annulus must remain inside a gauge slice with regulator-uniform geometry.

16.3 The nonlinear completion-band theorem

Program Requirement 16.4 (Uniform local completion band). Assume theorems 14.6 and 15.4. Prove that there exist a fixed integer r_* , band projections $\Pi_{B,n}^{(\sigma)}(u)$, and positive numbers $\Gamma_{\tau,n}$ such that:

- (CB1) $\text{rank } \Pi_{B,n}^{(\sigma)}(u) \leq r_*$ uniformly;
- (CB2) the purely auxiliary complement satisfies

$$(1 - \Pi) \mathcal{C}_{B,n}^{(\sigma)}(u) (1 - \Pi) \geq (E_{0,B,n}^{(\sigma)}(u) + \Gamma_{\tau,n}) (1 - \Pi), \quad \Gamma_{\tau,n} \rightarrow \infty; \quad (125)$$

- (CB3) the projections are norm-resolvent C^2 in the physical base variables and equivariant under every local exact symmetry, with derivative bounds strong enough for theorems 16.6 and 19.5;
- (CB4) in the trivial boundary/flux sector the physical fast ground band is rank one and generated by a smooth strictly positive real vector $\phi_{B,n}(u)$;
- (CB5) every other finite band degeneracy is classified by the boundary/topological sector, and any nonphysical degeneracy either decouples or is pushed to energy tending to infinity;
- (CB6) the band is the spectral band of the exact conditional fast form appearing in equation (110).

Proposition 16.5 (Berry triviality for a positive rank-one fast vacuum). *Let $u \mapsto \phi(u)$ be a C^1 family of normalized real vectors in a real form of a complex Hilbert space, with $\phi(u) > 0$ pointwise. Then the Abelian Berry one-form*

$$\mathcal{A}_{\text{Berry}} = i\langle \phi, d\phi \rangle$$

vanishes in the positive gauge. The Born–Huang tensor

$$\mathcal{G}_{\text{BH}}(X, Y) = \langle (1 - |\phi\rangle\langle\phi|)\nabla_X\phi, (1 - |\phi\rangle\langle\phi|)\nabla_Y\phi \rangle$$

need not vanish.

Proof. Differentiating $\langle \phi, \phi \rangle = 1$ gives $2\operatorname{Re}\langle \phi, d\phi \rangle = 0$. Because the family is real, the inner product is real, hence it is zero. Positivity fixes the sign continuously and prevents an additional local phase choice. The orthogonal derivative may nevertheless be nonzero, which is precisely the Born–Huang tensor. $\square$

Program Requirement 16.6 (Berry and Born–Huang control). In addition to [theorem 5.7](#), for the physical rank-one completion band prove global compatibility of the positive gauge and the quantitative relative-form estimate

$$|q_{\text{BH},n}[\psi]| \leq \varepsilon_{\text{BH},n}(q_{\text{YM},n}[\psi] + \Lambda_*\|\psi\|^2), \quad \varepsilon_{\text{BH},n} \rightarrow 0. \quad (126)$$

For any unavoidable degenerate band, prove that the matrix-valued Berry curvature is gauge-equivalent to a scalar flat connection, is absorbed into already declared topological data, or belongs to a sector whose energy diverges. No additional finite-energy gauge field may survive.

Program Requirement 16.7 (Uniform conditional fast-sector Poincaré inequality). As an equivalent or alternative route to the spectral-band construction, prove uniformly in B, n, u, σ that

$$\operatorname{Var}_{\mu_{B,n,u}^{\text{aux}}}(f) \leq \Gamma_{\tau,n}^{-1} \left(c_{B,n,u}^{(\sigma)}[f] - E_{0,B,n}^{(\sigma)}(u)\|f\|^2 \right), \quad \Gamma_{\tau,n} \rightarrow \infty, \quad (127)$$

for f orthogonal to the complete fast ground band. Together with compact resolvent this gives [equation \(125\)](#); conversely, [equation \(125\)](#) gives [equation \(127\)](#) by the spectral theorem.

Program Requirement 16.8 (First concrete fast-sector milestone for $SU(2)$). Specialize to $G = SU(2)$, the trivial magnetic and center sector, one fixed physical block, and the frozen scaling [\(21\)](#). Prove uniformly in the physical base field u that

$$\operatorname{rank} \Pi_{B,n}(u) = 1, \quad E_{1,B,n}(u) - E_{0,B,n}(u) \geq \Gamma_{\tau,n}, \quad \Gamma_{\tau,n} \rightarrow \infty, \quad (128)$$

and establish gauge-covariant derivative estimates

$$\sup_u \|\nabla_u^j \Pi_{B,n}(u)\| \leq C_j \Gamma_{\tau,n}^{\theta_j} \quad (0 \leq \theta_j < 1) \quad (129)$$

strong enough to make every Berry/Born–Huang and cross-form correction $o(1)$ relative to the four-dimensional physical form. The positive rank-one eigenfunction must be compatible on gauge-slice overlaps. This theorem is the first go/no-go test of whether the DTT completion sector gives genuinely new analytic leverage.

Proof Audit 16.9 (Status of the local band). The proved tangent estimate and abstract IMS lemma do not establish [theorems 16.4](#) and [16.7](#). Missing inputs include the regulator-uniform nonlinear normal form, gauge-slice Jacobians, all holonomy strata, exclusion of additional wells, and uniform base-derivative bounds. The positive rank-one observation in [theorem 16.5](#) removes an Abelian Berry phase only after uniqueness and smoothness of the fast ground state have been proved; it does not control the Born–Huang potential. These remain the first decisive DTT-specific obligations.

17 Global completion band and gauge-compatible patching

A global low-energy band cannot be obtained by naively tensoring local projections because the gauge-invariant Hilbert space does not factor into independent block Hilbert spaces and neighboring completion terms overlap.

17.1 The commuting benchmark

Lemma 17.1 (Gap of a commuting tensor sum). *Let $C_j \geq 0$ act on $\mathcal{H}_j$, with simple ground energy zero and $C_j \geq \gamma_j$ on the ground-state orthogonal complement. On $\bigotimes_{j=1}^m \mathcal{H}_j$, set*

$$C = \sum_{j=1}^m I \otimes \cdots \otimes C_j \otimes \cdots \otimes I.$$

Then the tensor product of the local ground vectors is the unique ground vector and

$$C \geq \min_j \gamma_j$$

on its orthogonal complement.

Proof. The operators strongly commute and are simultaneously diagonal in tensor products of their eigenbases. Every non-ground product vector excites at least one factor, so its energy is at least the smallest local gap. Density and the spectral theorem give the operator inequality. $\square$

This lemma explains the desired mechanism but does not apply directly to the gauge-constrained overlapping-block theory.

17.2 Almost-projections

Lemma 17.2 (Rounding an almost-projection). *Let $A = A^*$ satisfy $0 \leq A \leq I$ and $\|A^2 - A\| \leq \varepsilon < 1/4$. Then*

$$\text{Spec}(A) \subseteq [0, \alpha_-] \cup [\alpha_+, 1], \quad \alpha_{\pm} = \frac{1 \pm \sqrt{1 - 4\varepsilon}}{2}.$$

The spectral projection $\Pi = \mathbf{1}_{[\alpha_+, 1]}(A)$ satisfies

$$\|A - \Pi\| \leq \alpha_-.$$

If a group commutes with A , it commutes with Π .

Proof. For $\lambda \in [0, 1]$, the bound $|\lambda^2 - \lambda| \leq \varepsilon$ excludes the open interval (α_-, α_+) . The functional-calculus estimate is then the maximum distance from λ to the nearest value 0 or 1. The symmetry statement follows from Borel functional calculus. $\square$

This lemma is useful if products or averaged products of local band projections can be shown to be almost idempotent.

Program Requirement 17.3 (Gauge-compatible global completion band). Assume theorems 14.6, 16.4 and 16.6. For a bounded-overlap cover $\mathfrak{B}_n$ by fixed physical blocks, construct a global equivariant projection Π_n^{cb} and an isometry

$$J_n : \mathcal{H}_n^{4D} \longrightarrow \Pi_n^{\text{cb}} \mathcal{H}_n^{\text{cb}}$$

such that:

- (GB1) Π_n^{cb} commutes with gauge projection, spatial symmetries, inner translation, and the vacuum projection;
 (GB2) the high completion block satisfies

$$(1 - \Pi_n^{\text{cb}})H_n^{\text{cb}}(1 - \Pi_n^{\text{cb}}) \geq (\Gamma_{\tau,n} - C_*)(1 - \Pi_n^{\text{cb}}), \quad \Gamma_{\tau,n} - C_* \longrightarrow \infty; \quad (130)$$

- (GB3) the vacuum is contained exactly,

$$\Pi_n^{\text{cb}}\Omega_n^{\text{cb}} = \Omega_n^{\text{cb}}; \quad (131)$$

- (GB4) the range is locally generated by the coherent fast band and is unitarily identified with a four-dimensional gauge-invariant Hilbert space up to an explicitly controlled error tending to zero;
 (GB5) local band overlaps, Gauss-law correlations, boundary modes, Berry/Born–Huang terms, and topological labels are controlled by a gauge-compatible martingale, conditional-expectation, or quasi-adiabatic patching theorem rather than by tensor-factorization assumptions.

Proof Audit 17.4 (Global patching frontier). Local spectral bands do not automatically give a global band. The proof must control overlaps, boundary modes, Gauss-law correlations, and the possibility that local ground bands have inconsistent topological labels. This is the first local-to-global theorem in the redesigned program.

18 Adiabatic Schur–Feshbach elimination

This section proves the abstract elimination theorem that will consume [theorem 17.3](#).

18.1 A block-operator theorem

Theorem 18.1 (Spectral Feshbach reduction below a transverse band). *Let $H = H^*$ on $\mathcal{H} = P\mathcal{H} \oplus Q\mathcal{H}$. Assume $P\text{Dom } H \subseteq \text{Dom } H$, write*

$$A = PHP, \quad B = PHQ, \quad D = QHQ,$$

and suppose B extends to a bounded operator with $\|B\| \leq b$. Suppose $D \geq \Gamma Q$ for some $\Gamma > 0$. For real $z < \Gamma$, define

$$F_H(z) = A - zP - B(D - zQ)^{-1}B^*. \quad (132)$$

Then:

- (i) $H - z \geq 0$ if and only if $F_H(z) \geq 0$;
 (ii)

$$0 \leq B(D - zQ)^{-1}B^* \leq \frac{b^2}{\Gamma - z}P; \quad (133)$$

- (iii) whenever $F_H(z)$ is invertible, the block inverse is

$$(H - z)^{-1} = \begin{pmatrix} F^{-1} & -F^{-1}B(D - z)^{-1} \\ -(D - z)^{-1}B^*F^{-1} & (D - z)^{-1} + (D - z)^{-1}B^*F^{-1}B(D - z)^{-1} \end{pmatrix}, \quad (134)$$

where $F = F_H(z)$, and

$$\|Q(H - z)^{-1}\| \leq \frac{1}{\Gamma - z} + \frac{b}{\Gamma - z}\|F_H(z)^{-1}\| \quad (135)$$

$$+ \frac{b^2}{(\Gamma - z)^2}\|F_H(z)^{-1}\|. \quad (136)$$

Proof. For $x \in P\mathcal{H}$ and $y \in Q \operatorname{Dom} H$, completion of the square gives

$$\begin{aligned} \langle x + y, (H - z)(x + y) \rangle &= \langle x, F_H(z)x \rangle \\ &\quad + \left\langle y + (D - z)^{-1}B^*x, (D - z)[y + (D - z)^{-1}B^*x] \right\rangle. \end{aligned}$$

This proves (i). Functional calculus gives $(D - z)^{-1} \leq (\Gamma - z)^{-1}Q$, proving (ii). Solving the two block equations yields equation (134). The lower row contains a first block of norm at most $b(\Gamma - z)^{-1}\|F^{-1}\|$ and a second block of norm at most $(\Gamma - z)^{-1} + b^2(\Gamma - z)^{-2}\|F^{-1}\|$. The triangle inequality gives equation (135). $\square$

Corollary 18.2 (Low-block stability). *If $A \geq \Delta P$ and*

$$\frac{b^2}{\Gamma - z} < \Delta - z,$$

then $H \geq z$. In particular, if $\Gamma \rightarrow \infty$ while b remains bounded, the low spectrum converges to that of A on every fixed compact energy interval.

Proof. By (133),

$$F_H(z) \geq \left(\Delta - z - \frac{b^2}{\Gamma - z} \right) P > 0.$$

Apply theorem 18.1. $\square$

Theorem 18.3 (Quadratic-form Feshbach identity). *Let q be a densely defined closed semi-bounded quadratic form on $\mathcal{H} = P\mathcal{H} \oplus Q\mathcal{H}$, and assume*

$$Q := \operatorname{Dom} q = PQ \oplus QQ.$$

For real z , set $q_z = q - z\|\cdot\|^2$ and write, using the polarized sesquilinear forms associated with the same symbols,

$$q_z[p + r] = a_z[p] + d_z[r] + 2 \operatorname{Re} b[p, r].$$

Assume d_z is closed and strictly positive on QQ , and that for each $p \in PQ$ the functional $r \mapsto b[p, r]$ is continuous in the Hilbert norm $d_z[r]^{1/2}$. Let $K_z p \in QQ$ be the unique Riesz representative satisfying

$$d_z[K_z p, r] = b[p, r] \quad (r \in QQ), \quad (137)$$

and define the Feshbach form

$$f_z[p] = a_z[p] - d_z[K_z p]. \quad (138)$$

Then

$$q_z[p + r] = f_z[p] + d_z[r + K_z p]. \quad (139)$$

Consequently $q_z \geq 0$ if and only if $f_z \geq 0$. If $a_z[p] \geq 0$ and

$$d_z[K_z p] \leq \eta a_z[p] \quad (0 \leq \eta < 1), \quad (140)$$

then $f_z[p] \geq (1 - \eta)a_z[p]$.

Proof. By equation (137),

$$d_z[r + K_z p] = d_z[r] + 2 \operatorname{Re} b[p, r] + d_z[K_z p].$$

Subtracting the final term gives equation (139). For fixed p , the right side is minimized at $r = -K_z p$, proving the equivalence. The last assertion follows directly from equations (138) and (140). $\square$

Remark 18.4 (Why the form theorem is the operative one). For a gauge Schrödinger Hamiltonian the block PHQ need not be bounded. The natural small parameter is the norm of the cross form between the slow and fast form spaces. The bounded-operator theorem remains a useful benchmark, but theorem 18.3 is the correct interface for the exact regulator.

18.2 Vacuum compatibility

If P contains the vacuum, the Feshbach operator carries the vacuum exactly. Strict positivity must be tested on the orthogonal complement of the transported vacuum line, just as in theorem 12.3. The completion-band construction improves the situation because $P = \Pi_n^{\text{cb}}$ is a spectral band designed to contain the vacuum, rather than a multiplication window that cuts through its positive wavefunction.

Program Requirement 18.5 (Uniform form-level adiabatic decoupling). Assume theorem 17.3. Prove that Π_n^{cb} preserves the form domain of H_n^{cb} and decompose, for $z \leq E$,

$$q_{n,z}^{\text{cb}}[p+r] = a_{n,z}[p] + d_{n,z}[r] + 2 \operatorname{Re} b_n[p, r], \quad p \in \Pi_n^{\text{cb}} \mathcal{Q}_n^{\text{sl}}, \quad r \in (1 - \Pi_n^{\text{cb}}) \mathcal{Q}_n^{\text{sl}}.$$

The high form must obey

$$d_{n,z}[r] \geq (\Gamma_{\tau,n} - E - C_*) \|r\|^2, \quad \Gamma_{\tau,n} \rightarrow \infty. \quad (141)$$

Let $K_{n,z}p$ be the form representer of equation (137). Prove the normalized cross-form estimate

$$d_{n,z}[K_{n,z}p] \leq \varepsilon_n(E)(a_{n,z}[p] + C_E \|p\|^2), \quad \varepsilon_n(E) \rightarrow 0, \quad (142)$$

uniformly for $z \in [-1, E]$, including the vacuum and all boundary/flux sectors. Define the effective closed form

$$f_{n,z}^{\text{eff}}[p] = a_{n,z}[p] - d_{n,z}[K_{n,z}p] \quad (143)$$

and transport it through J_n to the four-dimensional Hilbert space. Prove strong-resolvent convergence below E , and norm-resolvent convergence whenever the stated compactness and uniform resolvent bounds permit it:

$$(H_n^{\text{cb}} - z)^{-1} - J_n(H_n^{\text{eff}} - z)^{-1} J_n^* \longrightarrow 0. \quad (144)$$

The vacuum must satisfy $\Pi_n^{\text{cb}} \Omega_n^{\text{cb}} = \Omega_n^{\text{cb}}$ exactly and be transported to the unique zero vector of the effective form.

19 The effective four-dimensional gauge theory

19.1 Exact gauge invariance after auxiliary integration

At the Euclidean level, define the finite-block effective weight by integrating completion variables:

$$\exp[-S_{B,n}^{\text{eff}}(U)] = \int_{\mathcal{Y}_{B,n}(U)} \exp[-S_{B,n}^{\text{cb}}(U, y)] \, d\mu_{B,n,U}^{\text{aux}}(y). \quad (145)$$

Proposition 19.1 (Gauge invariance of the effective action). *If the auxiliary measure and completion-band action are gauge invariant, then $S_{B,n}^{\text{eff}}$ is gauge invariant. In particular, its local expansion can involve only gauge-invariant closed-loop or curvature operators; an open-link mass term is excluded.*

Proof. For a physical gauge transformation g , change variables $y \mapsto g \cdot y$ in (145). Invariance of the measure and action gives

$$e^{-S_{B,n}^{\text{eff}}(g \cdot U)} = e^{-S_{B,n}^{\text{eff}}(U)}.$$

A function of a single open link cannot be invariant under independent transformations at its two endpoints unless it is constant. Therefore nonconstant local terms must be built from closed holonomies or equivalent gauge-invariant curvature data. $\square$

This argument excludes an explicit gauge-boson Proca mass. It does not exclude a spectral mass gap in the gauge-invariant Hamiltonian.

19.2 Quasi-local expansion

A transverse band with exponential completion-sector clustering should imply a linked-cluster expansion

$$S_n^{\text{eff}}(U) = \frac{1}{g_{R,n}^2} \sum_{p \in \mathbb{R}_{a_n}^4} \ell_p(U_p) + \sum_{X \in \mathbb{R}_{a_n}^4} \Phi_{n,X}(U), \quad (146)$$

where each $\Phi_{n,X}$ is gauge invariant and

$$\sum_{X \ni 0} e^{\mu \text{diam}(X)/a_n} \|\Phi_{n,X}\|_{C^r} \leq \varepsilon_n, \quad \varepsilon_n \rightarrow 0. \quad (147)$$

The plaquette coefficient is fixed by a renormalization condition rather than by an assumed mass scale.

19.3 The extensive closure term and its local polymer norm

The pointwise limit $a_{t,n} \kappa_{H,n} \rightarrow 0$ controls the coefficient of one local closure term but does not control the sum over the growing spatial volume. The relevant quantity is a local interaction norm. For a translation-covariant family of gauge-invariant interactions $\Phi = \{\Phi_X\}$, define

$$\|\Phi\|_{\mu,r}^{\text{loc}} = \sup_x \sum_{X \ni x} e^{\mu \text{diam}(X)/a_n} \|\Phi_X\|_{C^r}. \quad (148)$$

This norm is volume independent when the interaction is local or exponentially summable.

Proof Audit 19.2 (Why a vanishing bare coefficient is insufficient). Even if $a_{t,n} \kappa_{H,n} \rightarrow 0$, the total closure action contains one term per spatial site. Free-energy densities, cumulants, and induced interactions can therefore remain nonzero. Decoupling must be proved in a norm such as (148), after integrating the fast variables and including all connected cumulants.

Program Requirement 19.3 (Volume-uniform closure decoupling). Let $\Phi_{n,X}^{\text{aux}}$ denote the connected four-dimensional interactions induced by the holonomy closure and the remaining fast completion variables. Prove, for some $\mu > 0$ and finite r ,

$$\|\Phi_n^{\text{aux}}\|_{\mu,r}^{\text{loc}} = \sup_x \sum_{X \ni x} e^{\mu \text{diam}(X)/a_n} \|\Phi_{n,X}^{\text{aux}}\|_{C^r} \rightarrow 0. \quad (149)$$

The estimate must be uniform in spatial volume and must include Berry, Born–Huang, holonomy, and overlap contributions. This is stronger than coefficientwise convergence and is the required form of auxiliary irrelevance.

Program Requirement 19.4 (Universality–coercivity scaling window). For the frozen sequences equations (15) and (21), prove simultaneously that

$$\Gamma_{\tau,n} \rightarrow \infty, \quad \varepsilon_{\text{adiab},n} \rightarrow 0, \quad \varepsilon_{\text{aux},n}^{\text{eff}} \rightarrow 0. \quad (150)$$

Here $\Gamma_{\tau,n}$ is the exact nonlinear fast-sector gap, $\varepsilon_{\text{adiab},n}$ is the normalized cross-form error in equation (142), and $\varepsilon_{\text{aux},n}^{\text{eff}}$ controls all induced four-dimensional interactions. The same fixed sequence $\kappa_{H,n}$ must satisfy all three limits; it may not be retuned after inspecting the effective action.

Program Requirement 19.5 (Controlled four-dimensional effective action). Starting from theorems 14.6, 16.4, 16.6, 17.3, 18.5, 19.3 and 19.4, prove equations (146) and (147) and an operator comparison

$$H_n^{\text{eff}} = H_n^{\text{KS/YM}} + R_n, \quad |\langle \psi, R_n \psi \rangle| \leq \varepsilon_n \left(\langle \psi, H_n^{\text{KS/YM}} \psi \rangle + \Lambda_* \|\psi\|^2 \right). \quad (151)$$

Prove that every auxiliary-induced relevant or marginal operator is either forbidden by exact gauge, reflection, and Euclidean symmetries or absorbed into the ordinary Yang–Mills coupling and vacuum-energy normalization. In particular, a Proca term is forbidden, a CP-odd theta term is absent in the reflection-symmetric regulator, and all Berry/Born–Huang contributions obey the relative bound [equation \(126\)](#). All remaining auxiliary operators must be irrelevant and satisfy $\varepsilon_n \rightarrow 0$.

Proof Audit 19.6 (Universality–coercivity compatibility). The auxiliary sector is useful only if it supplies a transverse band while disappearing from the four-dimensional universality class. Making $\kappa_{H,n}$ large can improve the band but can also induce large effective interactions. A complete proof must exhibit a scaling window in which both (125) and (147) hold. This compatibility is not established by the finite-regulator construction.

Part III

The open nonabelian and continuum completion program

20 Gauge-fixed block geometry

The completion-band theorem removes transverse auxiliary excitations. It does not control the physical τ -constant gauge field, which remains ordinary four-dimensional Yang–Mills near the coherent stratum. The next task is therefore a local gauge-rigidity theorem for the physical field.

20.1 A fixed-combinatorial-block flatness estimate

Let d_G be the geodesic distance of the fixed bi-invariant metric on G .

Lemma 20.1 (Defect controls group distance). *There is a constant $C_\rho < \infty$ such that*

$$d_G(g, e)^2 \leq C_\rho \ell_\rho(g) \quad (g \in G). \quad (152)$$

Proof. Near the identity, [theorem 7.1](#) gives $\ell_\rho(\exp X) = \frac{1}{2}\|X\|^2 + O(\|X\|^4)$, while $d_G(\exp X, e)^2 = \|X\|^2 + O(\|X\|^4)$. Hence the ratio is bounded in a neighborhood of the identity. On the complement of a smaller neighborhood, ℓ_ρ has a strictly positive minimum and d_G is bounded by compactness. Combining the two regions proves the claim. $\square$

Theorem 20.2 (Tree-gauge flatness on a finite cubical block). *Let B be a finite simply connected cubical complex with fixed boundary trivialization and let U be a G -valued lattice connection in the trivial flux sector. Then there is a gauge transformation g , equal to the identity on a chosen boundary root, such that*

$$\sum_{e \in B} d_G\left(g_{s(e)} U_e g_{t(e)}^{-1}, e\right)^2 \leq C_{B,\rho} \sum_{p \in B} \ell_\rho(U_p). \quad (153)$$

Proof. Choose a spanning tree $\mathcal{T}$ in the one-skeleton rooted at the fixed boundary vertex. Define g recursively so that every transformed tree edge equals the identity. For a non-tree edge e , the edge together with the two tree paths to the root forms a based loop γ_e . Since the cubical complex is simply connected, γ_e is the boundary of a finite plaquette two-chain. After tree gauge, the transformed value on e equals the ordered product of the plaquette holonomies in one such filling, with conjugations that do not affect the bi-invariant distance or ℓ_ρ .

For a bi-invariant geodesic distance,

$$d_G(g_1 \cdots g_m, e) \leq \sum_{j=1}^m d_G(g_j, e).$$

Cauchy–Schwarz and theorem 20.1 therefore give

$$d_G(U_e^g, e)^2 \leq m_e C_\rho \sum_{p \in \Sigma_e} \ell_\rho(U_p),$$

where m_e is the number of plaquettes in the chosen filling Σ_e . Sum over the finitely many non-tree edges and absorb the maximum overlap and filling size into $C_{B,\rho}$. Tree edges contribute zero. $\square$

The constant in (153) may grow rapidly when the number of lattice cells in a fixed physical block tends to infinity. The theorem is therefore a finite-combinatorial result, not yet a scale-uniform lattice Uhlenbeck theorem.

Program Requirement 20.3 (Scale-uniform lattice Uhlenbeck estimate). Fix a physical block side $R > 0$. For the refined blocks $B_{R,n}$, prove a gauge choice and a discrete Sobolev norm $\|\cdot\|_{W_{a_n}^{1,2}}$ such that, whenever the plaquette energy is below a fixed threshold,

$$\|A^g\|_{W_{a_n}^{1,2}(B_{R,n})}^2 \leq C_R \sum_{p \in B_{R,n}} a_n^4 |F_p|^2, \quad (154)$$

with C_R independent of n , and with a corresponding estimate in every allowed boundary and magnetic-flux sector after subtracting the appropriate flat background. Prove uniform control of transition functions between the required local gauge charts.

This is the discrete gauge-fixing estimate needed for convexity and small-field RG bounds. Its continuum model is Uhlenbeck’s small-curvature gauge theorem [52]; a lattice proof must reproduce the relevant scale-uniform constants rather than cite the continuum statement formally.

21 Local gauge-invariant Poincaré inequalities

21.1 Ground-state-transformed local forms

Let $\hat{T}_n^{\text{eff}}$ be the ground-state transform of the effective four-dimensional transfer operator. Its Dirichlet form is

$$\mathcal{E}_n^{\text{eff}}(f, f) = \langle f, (I - \hat{T}_n^{\text{eff}})f \rangle_{L^2(\nu_n^{\text{eff}})}. \quad (155)$$

For a block B , let $\nu_{B,n}^\eta$ be the conditional measure inside B given an exterior configuration η , and let $\mathcal{E}_{B,n}^\eta$ be the corresponding conditional form. All functions are gauge invariant or are evaluated after a fixed local gauge slice with the Faddeev–Popov density included.

Definition 21.1 (Sectorwise local ground space). For boundary and flux label σ , let $\mathcal{G}_{B,n}^{(\sigma)}$ be the null space of the local effective generator after all exact gauge directions have been removed. It may have dimension larger than one. Let $P_{B,n}^{(\sigma)}$ be its orthogonal projection.

21.2 Near and far regions

Choose a smooth gauge-invariant partition of unity

$$\chi_{\text{near}}^2 + \chi_{\text{far}}^2 = 1$$

according to the physical plaquette energy. The near region is treated by theorem 20.3; the far region is treated by the positive potential and compactness.

Theorem Package 21.2 (Inputs for local block rigidity). The block proof requires a gauge-fixed analogue of log-concave Poincaré control, together with nonconvex and topological corrections; compare the role of convexity in Brascamp–Lieb inequalities [53]. Concretely, it requires:

- (L1) a scale-uniform gauge slice and Jacobian bound;
- (L2) a near-flat Hessian estimate on the orthogonal complement of gauge and ground-space directions;
- (L3) a positive far-region energy floor;
- (L4) an IMS partition with localization error smaller than the two local margins;
- (L5) uniform boundary-condition and flux-sector bookkeeping;
- (L6) control of completion-band corrections from theorem 19.5.

Program Requirement 21.3 (Block interruption–curvature rigidity). For one fixed physical block size R , prove that there is $C_R < \infty$, independent of n , the exterior condition η , and the allowed sector σ , such that

$$\|f - P_{B,n}^{(\sigma)} f\|_{L^2(\nu_{B,n}^\eta)}^2 \leq C_R a_{t,n}^{-1} \mathcal{E}_{B,n}^\eta(f, f). \quad (156)$$

The form must include electric transfer energy, physical curvature, completion-band response, and boundary terms. The ground space must be completely classified.

The factor $a_{t,n}^{-1}$ converts the dimensionless one-step Dirichlet form into physical energy units.

21.3 Abstract local-to-global variance decomposition

Lemma 21.4 (Martingale variance decomposition). *Let (Ω, ν) be a probability space and*

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_m$$

a filtration with $\mathcal{F}_0$ trivial and $\mathcal{F}_m$ complete. For $f \in L^2(\nu)$, set $f_j = \mathbb{E}[f \mid \mathcal{F}_j]$. Then

$$\text{Var}_\nu(f) = \sum_{j=1}^m \|f_j - f_{j-1}\|_2^2. \quad (157)$$

Proof. The martingale differences $d_j = f_j - f_{j-1}$ are mutually orthogonal in L^2 , and $f - \mathbb{E}f = \sum_j d_j$. Taking squared norms gives the identity. $\square$

A volume-uniform Poincaré theorem follows if every martingale difference can be charged to a bounded number of local Dirichlet forms with summable constants. Establishing this charging estimate in a gauge theory is a program requirement because Gauss constraints and overlapping Wilson loops prevent a naive tensor-product argument.

Program Requirement 21.5 (Gauge-compatible finite-size criterion). Construct a block filtration or alternating conditional-expectation scheme for which equation (156) implies

$$\text{Var}_{\nu_n^{\text{eff}}}(f) \leq C_{\text{glob}} a_{t,n}^{-1} \mathcal{E}_n^{\text{eff}}(f, f), \quad (158)$$

with C_{glob} independent of spatial volume and refinement. The proof must preserve gauge invariance, all declared flux sectors, and the completion-band projection.

22 Gauge-covariant multiscale renormalization

The ultraviolet coupling tends to zero, while the mass gap is an infrared statement. A complete proof must connect the two regimes by a controlled multiscale map. Merely writing the two-loop running coupling does not construct that map.

22.1 Block scales and fields

Fix an integer block factor $L \geq 2$. At RG step k , the lattice spacing is

$$a_{n,k} = L^k a_n.$$

Let Φ_k denote a gauge-fixed small-field coordinate on regular blocks and let $\mathfrak{X}_k$ denote polymers, i.e. unions of L^k -blocks. Write the effective density schematically as

$$\rho_k(\Phi_k) = \exp[-S_k^{\text{rel}}(\Phi_k)] \sum_{\{X_j\} \text{ disjoint}} \prod_j K_k(X_j, \Phi_k), \quad (159)$$

where S_k^{rel} contains the finite-dimensional relevant and marginal coordinates and K_k is a connected polymer activity.

22.2 Polymer norms

For an activity differentiable to order r , define a representative norm

$$\|K_k\|_{h,\kappa} = \sup_X A^{|X|_k} e^{\kappa \text{diam}_k(X)} \sup_{\Phi_k \in \mathcal{R}_k(X)} \sum_{j=0}^r \frac{h^j}{j!} \|D^j K_k(X, \Phi_k)\|, \quad (160)$$

where $\mathcal{R}_k(X)$ is the regular-field domain and the large-field complement is controlled by a separate regulator. The exact norm used in a proof may differ, but it must simultaneously control derivatives, polymer size, large fields, gauge charts, and reflection.

22.3 One-step map

A block transformation consists of:

- (R1) gauge-covariant averaging of fine links to coarse links;
- (R2) local gauge fixing on regular blocks;
- (R3) Gaussian or heat-kernel integration of fluctuation fields;
- (R4) extraction of relevant and marginal local terms;
- (R5) reblocking and connected-cluster expansion;
- (R6) restoration of gauge-covariant notation and Ward identities.

Program Requirement 22.1 (One-step gauge-covariant RG theorem). Construct a Banach domain $\mathfrak{D}_k$ of effective actions containing the completion-band effective action of [theorem 19.5](#). Prove that one RG step is well defined and maps

$$\mathcal{R}_k : \mathfrak{D}_k \longrightarrow \mathfrak{D}_{k+1}$$

with the following properties:

- (RG1) exact local gauge invariance and preservation of boundary/flux sectors;
- (RG2) compatibility with outer-time reflection positivity or an equivalent transfer-operator representation at every scale;

(RG3) a running coupling relation

$$g_{k+1} = g_k + b_0(\log L)g_k^3 + b_1(\log L)g_k^5 + R_{g,k}, \quad |R_{g,k}| \leq Cg_k^7; \quad (161)$$

(RG4) contraction of auxiliary and other irrelevant coordinates,

$$\|c_{k+1}^{\text{irr}}\| \leq L^{-\omega}\|c_k^{\text{irr}}\| + Cg_k^p, \quad \omega > 0; \quad (162)$$

(RG5) a polymer estimate

$$\|K_{k+1}\|_{h_{k+1}, \kappa_{k+1}} \leq q\|K_k\|_{h_k, \kappa_k} + Cg_k^p, \quad q < 1; \quad (163)$$

(RG6) exponentially small large-field errors and uniform overlap estimates for gauge charts;

(RG7) preservation of the continuum Ward identities at the level of renormalized insertions.

All constants must be uniform for the number of steps needed to reach a fixed physical scale.

The Balaban program supplies major components of such a construction in small-field regimes [27–30]. The theorem above additionally requires compatibility with the completion-band effective action, the full diagonal regulator trajectory, and the infrared continuation used below.

23 The asymptotically free trajectory

23.1 Renormalized coupling condition

Fix a physical scale $\mu > 0$ and a gauge-invariant finite-volume observable $\mathcal{O}_\mu$, such as a normalized small Wilson-loop response or gradient-flow coupling. Define the renormalized coupling by

$$g_{R,n}^2(\mu) = \mathfrak{R}_\mu(\langle \mathcal{O}_\mu \rangle_n), \quad (164)$$

where $\mathfrak{R}_\mu$ is fixed independently of the desired mass gap.

23.2 Optional inner-spectrum matching functionals

The companion spectral-geometry paper proposes trace functionals of a protected inner operator as a way to organize common high-scale corrections. Such a quantity is mathematically meaningful only under an explicit trace-class hypothesis and cannot replace the four-dimensional renormalization condition (164). Let $L_{F,n} \geq 0$ be an auxiliary fast generator, let $P_{a,n}$ be a finite-rank or trace-compatible carrier projection, and let f be a positive rapidly decreasing profile. When trace class, define

$$\mathcal{G}_{a,n}(\mu) = \kappa_a \text{Tr}\left(P_{a,n}f(L_{F,n}/\mu^2)\right). \quad (165)$$

This is a matching diagnostic for how a common inner spectrum could enter several effective sectors. The physical pure-Yang–Mills coupling is still fixed by (164); $\mathcal{G}_{a,n}$ is not a prediction of its numerical value.

Program Requirement 23.1 (Irrelevance of auxiliary spectral coupling corrections). If functionals of the form (165) occur in the effective action, prove in the same local polymer norm used in [theorem 19.3](#) that their contribution is either absorbed into the ordinary Yang–Mills coupling and vacuum-energy normalization or tends to zero. In particular, any anomalous power or log-periodic term generated by an auxiliary spectral zeta function must disappear from the universal pure-Yang–Mills continuum limit unless it is independently derived as a property of that target theory.

Proof Audit 23.2 (Spectral traces are not a renormalization theorem). Functional calculus makes (165) well defined under trace-class assumptions, but it does not prove the Yang–Mills beta function, the existence of a renormalized trajectory, or regulator independence. Those remain the content of theorems 23.3, 24.2 and 31.1.

Program Requirement 23.3 (Nonperturbative ultraviolet trajectory). For every sufficiently small prescribed $g_R(\mu) > 0$, prove that there is a unique microscopic sequence of bare couplings and allowed counterterms such that

$$g_{R,n}(\mu) \longrightarrow g_R(\mu).$$

Derive, rather than assume, the asymptotic formula

$$g_n^2 = \frac{1}{2b_0 \log(a_n^{-1}/\Lambda_G)} \left[1 + O\left(\frac{\log \log(a_n^{-1})}{\log(a_n^{-1})}\right) \right] \quad (166)$$

and prove that the completion-band couplings lie in the stable irrelevant manifold of the RG transformation.

The two-loop sequence of theorem 7.2 is retained only as a reference parametrization. It becomes physically meaningful only after theorem 23.3 is proved.

23.3 Dimensional transmutation

Once a trajectory is constructed, define the RG invariant scale by

$$\Lambda_G = \mu \exp \left[- \int^{g_R(\mu)} \frac{dg}{\beta(g)} \right] \times \text{the chosen scheme normalization.} \quad (167)$$

No spectral mass is used to define Λ_G . The theorem must later show that the physical gap is a positive multiple of this independently defined scale.

24 Crossover to a massive infrared domain

Ultraviolet control alone does not imply a mass gap. A constructive proof needs a theorem connecting the weak-coupling trajectory to a domain in which a convergent infrared expansion gives exponential clustering.

- Definition 24.1** (Massive polymer domain). A scale- k effective action belongs to $\mathfrak{D}_{\text{mass}}$ if:
- (D1) its character or local transfer expansion has a unique vacuum sector;
 - (D2) the connected polymer activity satisfies a Kotecký–Preiss-type summability bound [57] with constant $q_* < 1$;
 - (D3) gauge-invariant insertions have exponentially summable truncated activities;
 - (D4) all auxiliary-induced relevant coordinates vanish or have been absorbed into ordinary Yang–Mills normalization;
 - (D5) the resulting correlation length in coarse lattice units is bounded uniformly.

Program Requirement 24.2 (Completion-assisted RG capture). There exist $L \geq 2$, an RG step number k_n , and constants $q_* < 1$, $m_* > 0$ such that

$$a_n L^{k_n} \longrightarrow \Lambda_G^{-1}, \quad (168)$$

and the scale- k_n effective action lies in $\mathfrak{D}_{\text{mass}}$ with constants independent of n . In addition,

$$\|c_{k_n}^{\text{aux}}\| \leq C a_n^\omega \quad (169)$$

for some $\omega > 0$.

24.1 Three-regime proof strategy

A credible proof must cover three regimes without a gap in scale.

- (C1) **Weak coupling.** Iterate theorem 22.1 while $g_k \leq g_{\text{pert}}$, controlling errors for $O(g_n^{-2})$ steps.
- (C2) **Intermediate coupling.** Cover the compact interval $[g_{\text{pert}}, g_*]$ by rigorous analytic bounds or validated interval computations for the finite-dimensional relevant RG map, with every discarded polymer tail bounded analytically.
- (C3) **Massive domain.** Prove that the endpoint satisfies the character/polymer convergence criterion defining $\mathfrak{D}_{\text{mass}}$.

Proof Audit 24.3 (The central Yang–Mills frontier). Theorem 24.2 is the deepest unresolved theorem in the paper. It is not a consequence of the transverse completion band. The band may improve locality and suppress auxiliary directions, but the transition from asymptotic freedom to a massive four-dimensional infrared theory must still be proved from nonabelian dynamics.

25 Massive-domain clustering

Program Requirement 25.1 (Uniform exponential clustering). Assume theorem 24.2. Prove that for gauge-invariant local observables A, B in the scale- k_n theory,

$$|\langle AB \rangle_{k_n} - \langle A \rangle_{k_n} \langle B \rangle_{k_n}| \leq C_{A,B} e^{-m_* d_{k_n}(A,B)}, \quad (170)$$

with $m_* > 0$ independent of n . Prove the same estimate for time-separated positive-time observables in a dense OS-cyclic core.

The physical mass scale associated with the coarse estimate is

$$\Delta_G = m_* \lim_{n \rightarrow \infty} (a_n L^{k_n})^{-1} = m_* \Lambda_G > 0. \quad (171)$$

Theorem 25.2 (Euclidean time decay implies a Hamiltonian gap). *Let $H \geq 0$ have a unique vacuum Ω , and let $\mathcal{C} \subseteq \Omega^\perp$ be dense. Suppose that for every $\psi \in \mathcal{C}$ and every $\varepsilon > 0$ there is $C_{\psi,\varepsilon} < \infty$ such that*

$$0 \leq \langle \psi, e^{-tH} \psi \rangle \leq C_{\psi,\varepsilon} e^{-(\Delta - \varepsilon)t} \quad (t \geq 0). \quad (172)$$

Then

$$\text{Spec}(H|_{\Omega^\perp}) \subseteq [\Delta, \infty).$$

Proof. Let μ_ψ be the spectral measure of H . If $\mu_\psi([0, \Delta - 2\varepsilon]) > 0$, then

$$\langle \psi, e^{-tH} \psi \rangle \geq e^{-(\Delta - 2\varepsilon)t} \mu_\psi([0, \Delta - 2\varepsilon]),$$

contradicting (172) for large t . Thus every $\psi \in \mathcal{C}$ has spectral support in $[\Delta, \infty)$. Density and continuity of spectral projections extend the conclusion to $\Omega^\perp$. $\square$

26 Uniform global gap

There are two equivalent endpoints for the finite-regulator infrared analysis: a global Poincaré inequality for the ground-state-transformed transfer operator, or uniform exponential decay on an OS-cyclic core.

Program Requirement 26.1 (Multiscale Poincaré inequality). Prove that there is $\Delta_G > 0$, independent of n and spatial volume, such that

$$\langle f, (I - \hat{T}_n^{\text{cb}})f \rangle_{L^2(\nu_n^{\text{cb}})} \geq (1 - e^{-a_{t,n}\Delta_G}) \text{Var}_{\nu_n^{\text{cb}}}(f) \quad (173)$$

for every gauge-invariant physical observable in the auxiliary-neutral sector. The proof must be obtained from theorems 21.3, 21.5, 24.2 and 25.1, with all constants tracked in physical units.

Proposition 26.2 (Poincaré inequality gives a transfer and Hamiltonian gap). *If (173) holds, then*

$$T_n^{\text{cb}} \leq e^{-a_{t,n}\Delta_G} I \quad \text{on } (\Omega_n^{\text{cb}})^\perp,$$

and therefore

$$H_n^{\text{cb}} \geq \Delta_G \quad \text{on } (\Omega_n^{\text{cb}})^\perp. \quad (174)$$

Proof. The ground-state transform is unitary and maps $(\Omega_n^{\text{cb}})^\perp$ to mean-zero functions. On that subspace, (173) gives

$$\langle f, \hat{T}_n^{\text{cb}} f \rangle \leq e^{-a_{t,n}\Delta_G} \|f\|^2.$$

Self-adjointness gives the operator bound. Applying the decreasing function $-a_{t,n}^{-1} \log$ by spectral calculus yields (174). $\square$

Corollary 26.3 (Punctured interruption response from the global gap). *Assume (174). For every $0 < \delta < \Delta_G$, every orthogonal projection P , and the associated vacuum-compatible gap operator, the Feshbach operator is nonnegative whenever its high block is invertible. On the punctured low space it inherits the spectral margin corresponding to $\Delta_G - \delta$. Thus the response inequality sought in theorem 12.11 becomes a consequence of the global constructive gap rather than an independent assumption.*

Proof. The Hamiltonian bound is equivalent to nonnegativity of the vacuum-corrected transfer-gap operator for every $\delta < \Delta_G$. The Schur–Feshbach criterion theorem 12.2 transfers this nonnegativity to the low Feshbach operator. Removing its transported vacuum kernel gives the punctured statement. Quantitative margins follow from the spectral distance between $e^{-a_t\delta}$ and $e^{-a_t\Delta_G}$. $\square$

Part IV

Continuum construction and intrinsic identification

27 A countable gauge-invariant observable core

A continuum construction must specify the observables whose correlations converge. It is not enough to prove weak convergence of bare link measures, because continuum Wilson loops and curvature fields require renormalization.

27.1 Wilson-loop generators

Let $\mathcal{L}_0$ be the countable set of piecewise linear loops in $\mathbb{R}^4$ whose vertices and affine parameters are rational. Let $\hat{G}_0$ be a countable set of finite-dimensional unitary representations separating

conjugacy classes of G . For $\gamma \in \mathcal{L}_0$ and $\pi \in \widehat{G}_0$, write

$$W_{\gamma,\pi}(A) = \text{Tr}_\pi \text{Hol}_A(\gamma).$$

At finite lattice spacing, choose a declared polygonal approximation γ_n and a renormalization factor $Z_{\gamma,\pi,n}$, including perimeter and cusp counterterms when required. Define

$$W_{\gamma,\pi}^{(n),R} = Z_{\gamma,\pi,n} \text{Tr}_\pi U_{\gamma_n}. \quad (175)$$

27.2 Bounded curvature generators

Let f range over a countable dense family of compactly supported rational test tensors, and let P range over invariant polynomials on $\mathfrak{g}$ with rational coefficients in a fixed basis. Formally write

$$F_{f,P} = \int_{\mathbb{R}^4} P(F_A(x)) f(x) \, d^4x.$$

Use bounded transforms

$$b(F_{f,P}) = F_{f,P} (1 + F_{f,P}^2)^{-1/2}. \quad (176)$$

Their lattice approximants are defined by small-loop curvature reconstruction and the same renormalization conditions used in the RG analysis.

Definition 27.1 (Countable Wilson–curvature core). Let $\mathfrak{W}_0$ be the unital $*$ -algebra over $\mathbb{Q} + i\mathbb{Q}$ generated by:

- (i) the renormalized Wilson variables (175);
- (ii) bounded curvature transforms (176);
- (iii) finite ordered products and rational outer-time translates;
- (iv) adjoints and finite linear combinations.

Let $\mathfrak{W}$ be a locally convex completion strong enough that the Euclidean functionals and positive-time products are continuous.

Program Requirement 27.2 (Core approximation and renormalization). Construct the renormalization factors and lattice approximants so that:

- (OC1) every generator has uniformly bounded moments of all orders needed for the OS topology;
- (OC2) the maps $\gamma \mapsto W_{\gamma,\pi}^R$ and $f \mapsto b(F_{f,P})$ are continuous in the declared topology;
- (OC3) small-loop reconstruction identifies the curvature generators;
- (OC4) $\mathfrak{W}_0$ is OS-cyclic in the continuum vacuum representation;
- (OC5) the construction is independent of the auxiliary completion regulator.

28 Convergence of Schwinger functionals

For $A \in \mathfrak{W}_0$, let

$$\mathcal{S}_n(A) = \langle \Omega_n^{\text{cb}}, A_n \Omega_n^{\text{cb}} \rangle$$

be the finite-regulator vacuum functional.

Lemma 28.1 (Reflection positivity is closed under pointwise limits). *Suppose $\mathcal{S}_n$ are reflection-positive functionals on a common positive-time $*$ -algebra and $\mathcal{S}_n(A) \rightarrow \mathcal{S}(A)$ for every algebra element appearing in finite OS quadratic forms. Then $\mathcal{S}$ is reflection positive.*

Proof. For a finite family $A_1, \dots, A_m$ and coefficients c_j , reflection positivity gives

$$\sum_{j,k} \overline{c_j} c_k \mathcal{S}_n(\Theta A_j A_k) \geq 0.$$

Every term converges, so the finite sum converges to the corresponding OS quadratic form for $\mathcal{S}$, which is therefore nonnegative. $\square$

Lemma 28.2 (Clustering gives vacuum uniqueness). *Let a translation-invariant OS functional reconstruct a vacuum representation. If connected correlations of a cyclic algebra satisfy*

$$\mathcal{S}(A\tau_x(B)) - \mathcal{S}(A)\mathcal{S}(B) \longrightarrow 0$$

as $|x| \rightarrow \infty$ in a spacelike or Euclidean direction, then the translation-invariant subspace is one dimensional.

Proof. Let P_0 be the projection onto translation-invariant vectors. The mean ergodic theorem gives weak convergence of spatial averages of $\tau_x(B)\Omega$ to $P_0B\Omega$. Clustering shows that every matrix element of this limit against the cyclic vectors $A\Omega$ equals $\langle A\Omega, \Omega \rangle \langle \Omega, B\Omega \rangle$. Hence $P_0B\Omega = \langle \Omega, B\Omega \rangle \Omega$. Cyclicity gives $P_0\mathcal{H} = \mathbb{C}\Omega$. $\square$

28.1 Restoration of four-dimensional Euclidean covariance

The exact Hamiltonian regulator has continuous outer time and a spatial lattice, so full $E(4)$ covariance is not automatic. It must be recovered through renormalized isotropy.

Program Requirement 28.3 (Isotropic $E(4)$ restoration). Prove along the renormalized trajectory that:

- (E41) spatial and temporal two-point normalizations converge to a common wave-function normalization;
- (E42) every allowed anisotropic relevant or marginal counterterm is either absent by reflection/cubic symmetry or is tuned by a renormalization condition fixed independently of the desired mass gap;
- (E43) lattice rotations, translations, and time translations converge on the Wilson-curvature core to a strongly continuous representation of $E(4)$;
- (E44) Schwinger functions transformed by an arbitrary compact set of Euclidean rotations converge uniformly to the rotated limiting functions;
- (E45) the reconstructed speed parameter is finite, nonzero, and absorbed by one final choice of physical units.

The proof must be compatible with reflection positivity and with the auxiliary-decoupling estimate [theorem 19.3](#).

Program Requirement 28.4 (Nontrivial four-dimensional OS limit). Along the trajectory of [theorems 23.3](#), [24.2](#) and [28.3](#), prove that

$$\mathcal{S}_G(A) = \lim_{n \rightarrow \infty} \mathcal{S}_n(A) \quad (A \in \mathfrak{W}_0) \quad (177)$$

exists and extends continuously to $\mathfrak{W}$. Prove:

- (OS1) normalization and positivity;
- (OS2) Euclidean translation and rotation covariance, including restoration from the lattice subgroup to $E(4)$;
- (OS3) reflection positivity with respect to outer Euclidean time;
- (OS4) permutation symmetry, regularity, and temperedness of the corresponding distributions;
- (OS5) locality of the reconstructed gauge-invariant fields;
- (OS6) clustering and a unique vacuum;
- (OS7) nontriviality of connected Wilson-loop or curvature correlations;
- (OS8) convergence independent of the admissible auxiliary scaling and closure-potential sequence.

Proof Audit 28.5 (Compactness is insufficient). Probability-measure tightness or pointwise convergence of a few Wilson loops would not prove [theorem 28.4](#). Renormalized insertions, OS regularity, locality, nontriviality, covariance restoration, and uniqueness must all be controlled on a separating algebra.

29 Comparison spaces, Mosco convergence, and gap transfer

The finite-regulator Hilbert spaces vary with n . A continuum gap theorem therefore needs explicit comparison maps.

Definition 29.1 (Identified Mosco convergence). Let $(\mathcal{H}_n, q_n, \Omega_n)$ be the finite-regulator physical theories and $(\mathcal{H}, q, \Omega)$ a candidate limit. Let

$$I_n : \mathcal{H}_n \rightarrow \widehat{\mathcal{H}}, \quad I : \mathcal{H} \rightarrow \widehat{\mathcal{H}}$$

be isometries into a common Hilbert space. We say q_n converges to q in the identified Mosco sense if:

(M1) whenever $I_{n_k} u_{n_k} \rightharpoonup Iu$ and the energies are bounded,

$$q[u] \leq \liminf_k q_{n_k}[u_{n_k}];$$

(M2) for every $u \in \text{Dom } q$, there are $u_n \in \text{Dom } q_n$ with

$$I_n u_n \rightarrow Iu, \quad q_n[u_n] \rightarrow q[u];$$

(M3) $I_n \Omega_n \rightarrow I\Omega$ strongly;

(M4) local observables and time translations converge on a common dense core.

Theorem 29.2 (Uniform gaps survive identified Mosco convergence). *Assume identified Mosco convergence and suppose*

$$q_n[u] \geq \Delta \left(\|u\|^2 - |\langle \Omega_n, u \rangle|^2 \right) \quad (178)$$

for one $\Delta > 0$ and all n . If $q[\Omega] = 0$, then

$$q[u] \geq \Delta \left(\|u\|^2 - |\langle \Omega, u \rangle|^2 \right). \quad (179)$$

Consequently the self-adjoint operator H associated with q satisfies

$$H\Omega = 0, \quad \text{Spec}(H|_{\Omega^\perp}) \subseteq [\Delta, \infty).$$

Proof. For $u \in \text{Dom } q$, take a recovery sequence u_n . Strong convergence of $I_n u_n$ and $I_n \Omega_n$ gives

$$\langle \Omega_n, u_n \rangle \rightarrow \langle \Omega, u \rangle.$$

Pass to the limit in (178) using convergence of the recovery energies. This gives (179). Since $q[\Omega] = 0$, the representation theorem gives $H^{1/2}\Omega = 0$. Restricting to $\Omega^\perp$ and applying the spectral theorem gives the spectral inclusion. $\square$

Program Requirement 29.3 (Comparison-space convergence). Construct the comparison Hilbert space and maps in theorem 29.1 for the regulator family. Prove identified Mosco convergence of the physical Hamiltonian forms, strong convergence of the transfer semigroups on the Wilson–curvature core, stability of vacuum orthogonality, and compatibility with the OS functional of theorem 28.4.

Once theorems 26.1 and 29.3 are proved, theorem 29.2 transfers the same Δ_G to the continuum.

30 Auxiliary invisibility

A gapped continuum limit of the enlarged regulator is not yet four-dimensional Yang–Mills. The auxiliary completion sector must disappear from local physics.

30.1 A general spectral decoupling lemma

Lemma 30.1 (Diverging auxiliary gap removes nonneutral low-energy states). *Let $H_n \geq 0$ commute with an orthogonal projection P_n , and suppose*

$$(1 - P_n)H_n(1 - P_n) \geq \Gamma_n(1 - P_n), \quad \Gamma_n \rightarrow \infty.$$

Then for every fixed $E < \infty$,

$$(1 - P_n)\mathbf{1}_{[0,E]}(H_n) = 0 \tag{180}$$

for all sufficiently large n .

Proof. Since P_n commutes with H_n , the two subspaces reduce the spectral calculus. The spectrum of the restriction to $(1 - P_n)\mathcal{H}_n$ lies in $[\Gamma_n, \infty)$. If $\Gamma_n > E$, the spectral projection onto $[0, E]$ vanishes there. $\square$

Applied to nontrivial inner-time character sectors, this lemma would show that fixed-energy excitations are eventually auxiliary neutral. It does not by itself remove neutral holonomy observables. On any linearized real comparison sector carrying the canonical positive-energy complex structure J_n , a second diagnostic is the antilinear leakage

$$\beta_{J_n}(S_n(t)) = \frac{1}{2}(S_n(t) + J_n S_n(t) J_n), \quad Q_{n,t}^{\text{leak}} = \beta_{J_n}(S_n(t))^* \beta_{J_n}(S_n(t)). \tag{181}$$

It vanishes exactly when the reduced real evolution preserves the selected complex structure. This is a phase-compatibility diagnostic, not a Hamiltonian mass-gap observable.

Program Requirement 30.2 (Auxiliary invisibility). Prove all of the following.

(AI1) **Nonneutral divergence:** for every nontrivial inner character χ ,

$$H_n^{\text{cb}}|_{\mathcal{H}_{n,\chi}} \geq \Gamma_{\tau,n}, \quad \Gamma_{\tau,n} \rightarrow \infty.$$

(AI2) **Holonomy concentration:** for every fixed physical compact region O ,

$$\sup_{x \in O} \langle \ell_\rho(H_x) \rangle_n \rightarrow 0.$$

(AI3) **Scalarization:** for every bounded neutral auxiliary observable B_n localized over O ,

$$\langle A_1[B_n - \langle B_n \rangle_n I]A_2 \rangle_n \rightarrow 0$$

for all physical-core observables A_1, A_2 .

(AI4) **Completion-band holonomy disappearance:** the Kato curvature of every removable completion band obeys the no-extra-gauge alternatives in [theorem 5.7](#); no new finite-energy $U(m)$ connection survives.

(AI5) **Antilinear leakage disappearance:** for every compact region O and fixed $T < \infty$, the localized version of (181) satisfies

$$\sup_{0 \leq t \leq T} \|\chi_O \beta_{J_n}(S_n(t)) \chi_O\| \rightarrow 0,$$

or an explicitly declared local Hilbert–Schmidt or matrix-element substitute when operator norm is too strong.

(AI6) **Route independence:** physical correlations are independent of the admissible sequence $N_{\tau,n}, a_{\tau,n}, \kappa_{H,n}$, of auxiliary gauge fixing, and of the choice of equivalent fast-band trivialization.

- (AI7) **Local-net conclusion:** the continuum local algebra is generated entirely by four-dimensional Wilson-loop and curvature observables, and no winding, holonomy, Kaluza–Klein, or residual adiabatic-gauge particle sector survives.

Proof Audit 30.3 (Neutral does not mean invisible). The Reynolds projection of theorem 2.2 removes nontrivial inner characters but leaves invariant functions of auxiliary holonomy and relative configurations. Auxiliary neutrality is therefore only the first step. Scalarization and route independence are indispensable.

31 Intrinsic Yang–Mills identification

The limiting theory must earn the name pure Yang–Mills through its intrinsic properties. It is not enough to define a target functional to equal the limit.

Program Requirement 31.1 (Intrinsic Yang–Mills identification). For the continuum functional $\mathcal{S}_G$, prove:

- (YM1) the compact gauge group and its global form are the declared G ;
- (YM2) the physical local net is generated by renormalized Wilson loops and invariant curvature polynomials;
- (YM3) the continuum Ward identities and renormalized Schwinger–Dyson relations are those of pure Yang–Mills;
- (YM4) the ultraviolet normalization and beta function agree with the asymptotically free trajectory of theorem 23.3;
- (YM5) any retained carrier-band holonomy is exactly the declared G -connection, while every removable completion-band holonomy satisfies theorem 5.7;
- (YM6) every auxiliary spectral matching functional satisfies theorem 23.1;
- (YM7) no additional relevant or marginal interaction induced by the completion regulator survives;
- (YM8) the functional is universal under admissible changes of lattice action, completion-band closure potential, and auxiliary scaling;
- (YM9) the theory is nontrivial, local, clustering, and has a unique vacuum.

31.1 Small-loop curvature reconstruction

A concrete bridge between Wilson loops and curvature is required. For a small oriented rectangle $\gamma_{\mu\nu}(x, \varepsilon)$, the formal expansion is

$$\text{Hol}_A(\gamma_{\mu\nu}) = I + \varepsilon^2 F_{\mu\nu}(x) + O(\varepsilon^3).$$

The continuum theorem must establish a renormalized smeared version of

$$\varepsilon^{-2} \left(W_{\gamma_{\mu\nu}(x, \varepsilon)}^R - W_{\text{triv}} \right) \longrightarrow F_{\mu\nu}(x) \quad (182)$$

in the topology of bounded transforms or affiliated local operators. Perimeter and cusp renormalization, operator domains, and the order of $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ must be controlled.

31.2 An abstract core-identification theorem

Theorem 31.2 (Core equality implies OS unitary equivalence). *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be continuous reflection-positive Euclidean functionals on the same completed algebra $\mathfrak{W}$, with the same reflection and time translations. Assume $\mathfrak{W}_0$ is dense and OS-cyclic for both. If*

$$\mathcal{S}_1(A) = \mathcal{S}_2(A) \quad (A \in \mathfrak{W}_0),$$

then their OS Hilbert spaces are unitarily equivalent. The unitary preserves the vacuum, intertwines the time semigroups and Hamiltonians, and intertwines the represented core observables. In particular, the Hamiltonian spectra and mass gaps agree.

Proof. Continuity and density extend equality to all products ΘAB in the positive-time algebra. Therefore the OS seminorms and null spaces agree on the core. The map $[A]_1 \mapsto [A]_2$ is a well-defined isometry. Cyclicity gives dense range, so it extends to a unitary. Equality of translated classes intertwines the positive semigroups, and uniqueness of the nonnegative self-adjoint generator intertwines the Hamiltonians. The same quotient construction intertwines the represented observables. $\square$

This theorem is useful only when the target functional is independently constructed. The preferred Clay-facing route is [theorem 31.1](#), which identifies the DTT limit intrinsically.

32 Publication status and consolidated completion program

32.1 Consolidated program requirements

The open program is organized into six dependency blocks. Detailed formulations occur at the points of use; this table is the publication-facing ledger.

Block	Requirements	Decisive deliverable
Fast geometry	theorems 5.7, 14.3, 14.6, 15.4, 16.4, 16.6 and 16.7	A rank-one removable completion band, or an explicitly classified retained carrier band, with a diverging auxiliary gap and no unintended gauge holonomy.
Global elimination	theorems 17.3 and 18.5	Exact-vacuum global patching and a form-resolvent reduction to a four-dimensional low-energy Hamiltonian.
Auxiliary irrelevance	theorems 19.3 to 19.5 and 23.1	A volume-uniform local polymer bound showing that only ordinary Yang–Mills relevant/marginal data survive and that optional inner-spectrum corrections are absorbed or vanish.
Nonabelian dynamics	theorems 20.3, 21.3, 21.5, 22.1, 23.3, 24.2, 25.1 and 26.1	A weak-to-massive RG trajectory and a regulator-uniform physical Poincaré inequality.
Continuum construction	theorems 27.2, 28.3, 28.4 and 29.3	Nontrivial $E(4)$ -covariant OS data and gap-preserving comparison-space convergence.
Identification	theorems 30.2 and 31.1	Scalarization of all auxiliary observables and intrinsic pure-Yang–Mills universality.

32.2 Conditional endpoint theorem

Theorem 32.1 (Conditional DTT-to-Yang–Mills endpoint). *Suppose that theorems 5.7, 14.3, 14.6, 15.4, 16.4, 16.6, 16.7, 17.3, 18.5, 19.3 to 19.5, 20.3, 21.3, 21.5, 22.1, 23.1, 23.3, 24.2, 25.1, 26.1, 27.2, 28.3, 28.4, 29.3, 30.2 and 31.1 are proved for the frozen regulator family and every compact connected G with simple Lie algebra. Then theorem 1.2 holds.*

Proof. The completion-band and adiabatic theorems reduce the low-energy enlarged theory to the effective four-dimensional gauge theory with vanishing auxiliary corrections. The lattice flatness, block Poincaré, finite-size, RG, and clustering theorems imply the uniform finite-regulator lower bound (174). The observable-core and OS-limit theorems construct the nontrivial continuum vacuum theory. Identified Mosco convergence and theorem 29.2 transfer the gap. Auxiliary invisibility removes the completion sector, and intrinsic identification proves that the resulting local net and functional are pure Yang–Mills. Therefore the continuum Hamiltonian satisfies (3). $\square$

32.3 Exact status

The present manuscript does not prove theorem 32.1’s hypotheses. It proves the finite-regulator transfer theory, exact continuous-time completion Hamiltonian, exact interruption geometry, hard-window no-go theorem, corrected punctured Feshbach implication, explicit linear fast-scale divergence, the corrected operator Feshbach estimate, the quadratic-form Feshbach identity, the standard Kato protected-band theorem with the corrected intrinsic connection formula, and the abstract transfer theorems. The exact slow–fast disintegration, nonlinear completion band, Berry/Born–Huang control, four-dimensional RG crossover, global Poincaré estimate, continuum construction, auxiliary invisibility, and intrinsic Yang–Mills identification remain program requirements. Consequently (3) is not asserted as an unconditional theorem.

32.4 Shortest noncircular proof-production order

The recommended order is:

- Stage 0.** Freeze the exact Hamiltonian regulator and parameter sequence. The present edition completes the formal part of this stage; a proof implementation must still verify the orbit-type atlas and slow–fast disintegration theorems 14.3 and 14.6.
- Stage 1.** For $G = SU(2)$, one declared trivial-flux block, and the fixed scaling equation (21), prove theorems 5.7, 15.4, 16.4, 16.6 and 16.7 with a unique positive rank-one fast ground state and no unintended Kato holonomy.
- Stage 2.** Prove exact-vacuum global patching and form-level adiabatic elimination, theorems 17.3 and 18.5.
- Stage 3.** Prove volume-uniform closure decoupling, the nonempty universality–coercivity window, auxiliary spectral-correction irrelevance, and the quasi-local four-dimensional effective action, theorems 19.3 to 19.5 and 23.1.
- Stage 4.** Prove the scale-uniform lattice Uhlenbeck, block Poincaré, and gauge-compatible finite-size results, theorems 20.3, 21.3 and 21.5.
- Stage 5.** Construct the one-step RG map and nonperturbative ultraviolet trajectory, theorems 22.1 and 23.3.
- Stage 6.** Prove the crossover to a massive domain, exponential clustering, and the volume-uniform Poincaré inequality, theorems 24.2, 25.1 and 26.1.
- Stage 7.** Construct and renormalize the Wilson–curvature core, continuum OS functional, and comparison-space limit, theorems 27.2, 28.3, 28.4 and 29.3.

Stage 8. Prove auxiliary/Berry invisibility and intrinsic Yang–Mills universality, [theorems 30.2](#) and [31.1](#).

Stage 1 tests whether DTT supplies genuine new analytical leverage. Stage 6 is the deepest Yang–Mills step. The last two stages ensure that the resulting gapped theory is neither a compactified auxiliary theory nor an unidentified lattice universality class.

32.5 Falsifiers

The proposed route fails or requires redesign if any of the following occurs:

- (F1) the exact global Hamiltonian cannot be disintegrated into the proposed physical and fast forms with controlled cross terms;
- (F2) the transverse completion-band gap collapses under refinement;
- (F3) the holonomy closure term produces a surviving non-Yang–Mills relevant interaction;
- (F4) local completion bands cannot be patched compatibly with the Gauss constraint;
- (F5) Berry, Born–Huang, or other Kato-band holonomy survives as an additional finite-energy field;
- (F6) a carrier-band structure group is confused with the prescribed Yang–Mills gauge group, or an auxiliary $U(m)$ factor survives;
- (F7) an auxiliary spectral trace produces residual anomalous or log-periodic running incompatible with pure Yang–Mills;
- (F8) the effective action leaves the Yang–Mills universality class;
- (F9) the scale-uniform lattice flatness or block Poincaré estimates fail;
- (F10) the RG trajectory cannot be controlled through intermediate coupling;
- (F11) the continuum functional is trivial, nonlocal, or fails OS regularity;
- (F12) auxiliary holonomy or winding remains visible in local correlators;
- (F13) the limiting Ward identities or ultraviolet normalization differ from pure Yang–Mills.

These are mathematical failure modes, not rhetorical caveats.

32.6 Conclusion

The rigorous content of the DTT interpretation can now be summarized without shortcuts:

inner-time recurrence and covariant interruption
 → an exact global Schrödinger Hamiltonian with a proved slow–fast fiber decomposition
 → a transverse spectral band, if [theorem 16.4](#) holds
 → a four-dimensional effective gauge theory
 → a mass gap only after nonabelian RG coercivity and continuum identification.

Thus “mass from interrupted inner time” has a precise prospective meaning: after all hidden completion channels are allowed to relax, every physical nonvacuum sector must retain a positive outer-time response. The completion-band redesign identifies a concrete operator mechanism by which DTT might help establish that response. It does not replace the four-dimensional nonabelian proof. A complete solution requires the program requirements above to be replaced, one by one, by regulator-uniform proofs.

A Notation, scales, and operator roles

The same symbol is never used for the transfer kernel, the mixed-energy multiplication operator, and the auxiliary fiber difference. The principal notation is collected below.

Symbol	Meaning
a_n	physical spatial lattice spacing
$a_{t,n}$	outer Euclidean-time transfer step
$a_{\tau,n}$	auxiliary inner-time lattice spacing
$N_{\tau,n}$	number of auxiliary sites
$\beta_{\tau,n} = N_{\tau,n} a_{\tau,n}$	physical circumference of the auxiliary completion cycle
$\mathcal{X}_n$	one-slice product configuration space
$\mathcal{G}_n$	one-slice local gauge group
$\mathbf{P}_n = e^{-a_{t,n} \mathcal{L}_n}$	gauge-restricted heat semigroup
T_n	normalized diagnostic Strang transfer operator
$\mathbb{H}_{n,0}^{\text{cb}}, H_n^{\text{cb}}$	exact unshifted and ground-energy-shifted completion Hamiltonians
$T_n^{\text{cb}} = e^{-a_{t,n} H_n^{\text{cb}}}$	exact completion-band transfer semigroup
H_n	diagnostic Hamiltonian $-a_{t,n}^{-1} \log T_n$
$\Omega_n, \Omega_n^{\text{cb}}$	Perron–Frobenius vacuum vectors
$K_n = M_{\mathcal{X}_n}$	mixed-energy multiplication operator
$D_{\tau,V}^{\text{cov}}$	configuration-dependent covariant auxiliary difference
H_x	auxiliary holonomy at spatial site x
$\mathfrak{J}_{\tau,n}$	gauge-invariant inner-flow interruption functional
$G_{n,\delta}$	vacuum-corrected transfer-gap operator
$F_{\varepsilon,n}^{\text{vac}}(\delta)$	hard-window vacuum-compatible Feshbach operator
$c_{\tau,n}^{\text{kin}}, \alpha_n^{\text{mix}}, \kappa_{H,n}$	frozen kinetic, mixed-potential, and closure coefficients
$\mathcal{C}_{B,n}^{(\sigma)}(u)$	conditional fast Hamiltonian, pending the slow–fast disintegration theorem
$\Pi_{B,n}^{(\sigma)}(u), \Pi_n^{\text{cb}}$	local and global completion-band projections
H_n^{eff}	form-level adiabatically reduced four-dimensional effective Hamiltonian
$\mathcal{A}_{\text{Berry}}, \mathcal{G}_{\text{BH}}$	Berry connection and Born–Huang tensor of the fast band
$\mathfrak{W}_0$	countable renormalized Wilson–curvature core
$\mathcal{S}_G$	target continuum Euclidean functional
Δ_G	target positive physical mass gap

A.1 Dimensions and normalization

Natural units $\hbar = c = 1$ are used. The physical Hamiltonian has inverse-length dimension. A transfer threshold associated with a physical energy δ is

$$r_{n,\delta} = e^{-a_{t,n}\delta} = 1 - a_{t,n}\delta + O(a_{t,n}^2).$$

Accordingly, a regulator-uniform physical gap corresponds to a one-step transfer margin of order $a_{t,n}$, not to a fixed dimensionless gap below one.

A.2 Claim-status symbols

A boxed program requirement is not an assumption silently consumed by earlier results. The final conditional theorem explicitly lists every target it consumes. A proof audit records either a counterexample, a missing uniform estimate, or a required identification theorem.

B Global forms and flux labels for all simple Lie algebras

Every compact connected group with simple Lie algebra has the form

$$G = \tilde{G}/\Gamma', \quad \Gamma' \leq Z(\tilde{G}),$$

where $\tilde{G}$ is compact and simply connected. Then

$$\pi_1(G) = \Gamma', \quad Z(G) = Z(\tilde{G})/\Gamma'.$$

The possible simply connected centers are listed below.

Lie type	Simply connected group	Center
$A_\ell, \ell \geq 1$	$SU(\ell + 1)$	$\mathbb{Z}_{\ell+1}$
$B_\ell, \ell \geq 2$	$\text{Spin}(2\ell + 1)$	$\mathbb{Z}_2$
$C_\ell, \ell \geq 2$	$Sp(\ell)$	$\mathbb{Z}_2$
$D_\ell, \ell \geq 4, \ell \text{ even}$	$\text{Spin}(2\ell)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$D_\ell, \ell \geq 5, \ell \text{ odd}$	$\text{Spin}(2\ell)$	$\mathbb{Z}_4$
E_6	compact simply connected E_6	$\mathbb{Z}_3$
E_7	compact simply connected E_7	$\mathbb{Z}_2$
E_8, F_4, G_2	corresponding simply connected group	trivial

Because $H^2(T^3; \Gamma') \cong (\Gamma')^3$, every magnetic sector on the spatial torus is represented by three finite flux labels. In the untwisted magnetic sector, the three independent center-cut transformations give

$$\mathcal{C}_0 \cong Z(G)^3, \quad \chi \in \widehat{Z(G)^3}.$$

For a nontrivial magnetic cocycle, the allowed center transformations are the subgroup $\mathcal{C}_m$ preserving that cocycle. This is the sector convention used in section 7.5. The no-go theorem does not require any nontrivial flux: it applies in $(m, \chi) = (0, 1)$ for every global form.

C Finite-period trace states and the vacuum process

A finite periodic outer-time path integral represents a trace state. The vacuum state is obtained either from the two-sided ground-state-transformed Markov process or by an infinite-period limit taken at fixed regulator.

Proposition C.1 (Infinite-period convergence at fixed regulator). *Let T be a positive trace-class contraction with simple top eigenvalue 1, normalized eigenvector Ω , and remaining spectral radius $r < 1$. For bounded A , define*

$$\omega_M^{\text{tr}}(A) = \frac{\text{Tr}(T^M A)}{\text{Tr}(T^M)}.$$

Then

$$\omega_M^{\text{tr}}(A) \longrightarrow \langle \Omega, A \Omega \rangle, \tag{183}$$

and

$$\left| \omega_M^{\text{tr}}(A) - \langle \Omega, A \Omega \rangle \right| \leq 2 \|A\| r^{M-1} \text{Tr } T. \tag{184}$$

Proof. Choose an eigenbasis $\Omega, e_1, e_2, \dots$, with eigenvalues $1, t_1, t_2, \dots$. Then

$$\mathrm{Tr}(T^M A) = \langle \Omega, A\Omega \rangle + \sum_{j \geq 1} t_j^M \langle e_j, Ae_j \rangle,$$

and $\mathrm{Tr}(T^M) = 1 + \sum_{j \geq 1} t_j^M$. Since

$$\sum_{j \geq 1} t_j^M \leq r^{M-1} \sum_{j \geq 1} t_j \leq r^{M-1} \mathrm{Tr} T,$$

the estimate follows by elementary algebra and $|\langle e_j, Ae_j \rangle| \leq \|A\|$. $\square$

The bound need not be uniform as regulators are removed because r_n may approach one. Therefore the infinite-period limit is taken at each fixed regulator before the continuum limit, exactly as specified in [section 7.6](#).

D Projection-inertia Schur module

Lemma D.1 (Stationary hidden-variable elimination). *Let*

$$\mathcal{K} = \begin{pmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{pmatrix} = \mathcal{K}^*, \quad K_{yy} > 0,$$

and define

$$\Phi(x, y) = \frac{1}{2} \langle x \oplus y, \mathcal{K}(x \oplus y) \rangle.$$

For fixed x , the unique stationary point is

$$y_*(x) = -K_{yy}^{-1} K_{yx} x,$$

and

$$\Phi(x, y_*(x)) = \frac{1}{2} \langle x, (K_{xx} - K_{xy} K_{yy}^{-1} K_{yx}) x \rangle. \quad (185)$$

Moreover $\mathcal{K} \geq 0$ if and only if its Schur complement in [equation \(185\)](#) is nonnegative.

Proof. Differentiation in y gives $K_{yx} x + K_{yy} y = 0$. Completing the square yields

$$\begin{aligned} 2\Phi(x, y) &= \langle x, (K_{xx} - K_{xy} K_{yy}^{-1} K_{yx}) x \rangle \\ &\quad + \langle y + K_{yy}^{-1} K_{yx} x, K_{yy} (y + K_{yy}^{-1} K_{yx} x) \rangle. \end{aligned}$$

The second term is nonnegative and can be made zero, proving every assertion. $\square$

Proof Audit D.2 (Response analogy only). The Feshbach map [equation \(79\)](#) has the same block algebra as [equation \(185\)](#). In the language of [section 11](#), the high sector is the hidden completion channel and the punctured Feshbach operator is the response left after that channel attempts to restore inner-time coherence. The analogy supplies no positivity; indeed [theorem 1.1](#) shows that an incautious mixing formulation is inconsistent on the symmetry-protected vacuum line.

E Twisted Fourier and linear completion calculations

Let $\mathcal{V}$ be a finite-dimensional complex Hilbert space, $\mathcal{H} \in U(\mathcal{V})$, and $N \geq 2$. Define

$$(D_{\mathcal{H}}a)(r) = \frac{a(r+1) - a(r)}{a_\tau} \quad (0 \leq r < N-1),$$

$$(D_{\mathcal{H}}a)(N-1) = \frac{\mathcal{H}a(0) - a(N-1)}{a_\tau}.$$

Choose an orthonormal basis v_j with $\mathcal{H}v_j = e^{i\theta_j}v_j$. The vectors

$$a_{k,j}(r) = N^{-1/2} \exp\left(\frac{i(2\pi k + \theta_j)r}{N}\right) v_j$$

form an orthonormal basis and satisfy

$$D_{\mathcal{H}}a_{k,j} = a_\tau^{-1} \left(e^{i(2\pi k + \theta_j)/N} - 1 \right) a_{k,j}.$$

Consequently

$$D_{\mathcal{H}}^* D_{\mathcal{H}} a_{k,j} = \frac{4}{a_\tau^2} \sin^2\left(\frac{2\pi k + \theta_j}{2N}\right) a_{k,j}.$$

The kernel consists of constant sequences in the invariant subspace of $\mathcal{H}$. For $k=0$ and small nonzero θ , the eigenvalue is

$$\frac{\theta^2}{N^2 a_\tau^2} + O\left(\frac{\theta^4}{N^4 a_\tau^2}\right),$$

which is the near-trivial-holonomy obstruction.

E.1 Mean and relative modes

For an untwisted sequence ξ_r , the orthogonal decomposition

$$\xi_r = \bar{\xi} + \eta_r, \quad \sum_r \eta_r = 0,$$

separates the holonomy/mean mode from relative modes. The closure term in equation (115) controls the mean, while the cyclic difference controls η . This is the precise linear mechanism by which a holonomy closure potential repairs the zero mode without being mistaken for a physical gauge-boson mass.

F Supplementary band and Feshbach estimates

F.1 Contour estimates for band projections

Let C_1, C_2 be self-adjoint operators and let Γ be a contour contained in both resolvent sets. The resolvent identity gives

$$(z - C_1)^{-1} - (z - C_2)^{-1} = (z - C_1)^{-1}(C_1 - C_2)(z - C_2)^{-1}$$

when the difference is bounded. Therefore their Riesz projections satisfy

$$\|\Pi_1 - \Pi_2\| \leq \frac{|\Gamma|}{2\pi} \sup_{z \in \Gamma} \|(z - C_1)^{-1}\| \sup_{z \in \Gamma} \|(z - C_2)^{-1}\| \|C_1 - C_2\|.$$

For form perturbations the same conclusion follows after replacing operator-norm control by norm-resolvent control. A uniform contour distance is therefore the quantitative datum required to patch local completion bands.

F.2 Dangerous-cluster reduction

Theorem F.1 (Finite-rank reduction with an error margin). *Let $F = F^*$ be bounded, let Π be finite rank, $R = I - \Pi$, and assume*

$$RFR \geq \gamma R, \quad \gamma > 0.$$

Define

$$F_{\text{eff}} = \Pi F \Pi - \Pi F R (RFR)^{-1} R F \Pi.$$

Then $F \geq 0$ if and only if $F_{\text{eff}} \geq 0$. If a model matrix $M = M^*$ satisfies

$$M \geq \mu \Pi, \quad \|F_{\text{eff}} - M\| \leq \varepsilon < \mu,$$

then $F_{\text{eff}} \geq (\mu - \varepsilon)\Pi > 0$.

Proof. Apply completion of the square to the decomposition $\Pi\mathcal{H} \oplus R\mathcal{H}$. The norm estimate gives

$$F_{\text{eff}} \geq M - \|F_{\text{eff}} - M\|\Pi \geq (\mu - \varepsilon)\Pi.$$

□

The theorem is algebraic. A Yang–Mills application must still construct Π , prove the remainder floor γ , and obtain a model margin larger than its approximation error uniformly in the regulator.

F.3 Derivative of a spectral projection

Suppose $u \mapsto C(u)$ is differentiable in norm resolvent and a contour Γ isolates a band. Differentiating the Riesz formula gives

$$\partial_u \Pi(u) = \frac{1}{2\pi i} \int_{\Gamma} (z - C(u))^{-1} (\partial_u C(u)) (z - C(u))^{-1} dz.$$

Hence

$$\|\partial_u \Pi(u)\| \leq \frac{|\Gamma|}{2\pi} \sup_{z \in \Gamma} \|(z - C(u))^{-1}\|^2 \|\partial_u C(u)\|.$$

These derivatives are the Berry/connection terms produced when a completion band varies over the physical base. A complete effective-action theorem must prove that they are gauge covariant and irrelevant or absorbed into the Yang–Mills connection rather than introducing an additional physical field.

G RG verification ledger and reproducibility requirements

A proof of theorems 22.1 and 24.2 must expose every constant used to iterate the RG map. At minimum, each step should provide a certificate containing:

- (V1) the block factor L , field-domain radii, and large-field regulator;
- (V2) the gauge-fixing charts and quantitative Jacobian bounds;
- (V3) norms of fluctuation covariances and their spatial decay;
- (V4) extraction operators for all relevant and marginal gauge-invariant terms;
- (V5) the derivative bounds entering (160);
- (V6) the contraction factor $q < 1$, the irrelevant exponent ω , and the remainder in (161);
- (V7) boundary and flux-sector uniformity;
- (V8) reflection-positivity or transfer-operator compatibility;

- (V9) interval enclosures for every validated intermediate-coupling computation;
- (V10) a machine-readable list of rational upper and lower bounds from which the claimed inequalities can be rechecked independently.

Numerical evidence can guide the intermediate-coupling analysis but cannot replace a theorem. A validated computation is admissible only when analytic truncation errors and floating-point rounding are enclosed rigorously and when the finite calculation is linked to the infinite-dimensional polymer bounds.

H No-circularity and proof-status ledger

Table 3: Dependency and proof-status ledger.

Statement	Status	Exact role
Quaternionic rank-three readout	Proved under explicit hypotheses	Selects a local spatial carrier; no gap follows.
Spin-factor Lorentz cone	Conditional theorem under record-cone hypotheses	Reconstructs observer-level causal geometry; no matter dynamics follows.
Finite-regulator transfer and vacuum OS process	Proved	Supplies the diagnostic T_n, H_n, Ω_n and reflection positivity at fixed cutoff.
Exact completion	Proved at fixed cutoff	Supplies $\mathbb{H}_{n,0}^{\text{cb}}, H_n^{\text{cb}}, T_n^{\text{cb}}, \Omega_n^{\text{cb}}$ without a log–Strang identification.
Schrödinger regulator	Proved	Defines covariant interruption and rules out the bare Fourier shortcut.
Exact interruption identity and twisted spectrum		
Hard-window unpunctured mixing	Disproved	Fails on the transported vacuum line.
Punctured hard-window implication	Proved conditionally	Converts three explicit estimates into a finite-regulator gap.
Frozen linear completion stiffness	Proved	Both relative and holonomy tangent stiffnesses diverge for equation (21); no nonlinear band follows.
Slow-fast form disintegration	Theorem target	Connects the exact global Hamiltonian to the local completion fibers and their cross forms.
Nonlinear local completion band and conditional fast Poincaré	Theorem targets	First decisive DTT-specific uniform estimate.
Berry/Born–Huang control	Theorem target	Prevents the fast band from generating an additional finite-energy field.
Exact-vacuum global band and form-level adiabatic elimination	Theorem targets	Produce a four-dimensional low-energy effective theory without bounded-block assumptions.
Universality–coercivity scaling window and controlled effective action	Theorem targets	Must combine a diverging fast gap with vanishing non–Yang–Mills corrections.

Statement	Status	Exact role
Scale-uniform lattice flatness and block Poincaré	Theorem targets	Local nonabelian rigidity.
One-step RG and ultraviolet trajectory	Theorem targets	Construct asymptotically free continuum tuning.
RG crossover and massive clustering	Theorem targets	Central four-dimensional infrared theorem.
Uniform global Poincaré inequality	Theorem target	Gives the regulator-uniform physical gap.
Continuum OS limit and Mosco convergence	Theorem targets	Construct the continuum theory and transfer the gap.
Auxiliary invisibility and intrinsic YM identification	Theorem targets	Remove completion scaffolding and identify pure Yang–Mills.
Final mass-gap statement	Not asserted	Follows only after all preceding targets are proved.

H.1 No-circularity checklist

A claimed completion must answer each question affirmatively with a theorem and proof:

- (NC1) Is the physical scale fixed independently of the desired gap?
- (NC2) Is the bare trajectory derived from a renormalized observable rather than postulated?
- (NC3) Is the exact global Hamiltonian connected to the local fast fibers by a proved form disintegration?
- (NC4) Is the completion band proved uniformly rather than read from a bare Fourier frequency?
- (NC5) Is the transported vacuum handled exactly?
- (NC6) Are gauge, compact phase, observation filtering, and Lorentz symmetry kept distinct?
- (NC7) Are Berry and Born–Huang contributions proved absent, scalar, or irrelevant?
- (NC8) Is the effective action shown to remain in the Yang–Mills universality class?
- (NC9) Is the RG trajectory controlled through intermediate coupling?
- (NC10) Is the continuum functional nontrivial, local, and OS positive?
- (NC11) Are all auxiliary holonomy and winding observables removed from the physical net?
- (NC12) Is Yang–Mills identification intrinsic rather than definitional?

A negative answer is a missing theorem, not an editorial omission.

Manuscript form. This umbrella preprint preserves the full proof dependency graph. For specialist journal review, the completed finite-regulator, vacuum-obstruction, punctured-reduction, exact-Hamiltonian, and form-Feshbach results are extracted into the companion manuscript *Vacuum Obstructions and Quadratic-Form Feshbach Reduction in Auxiliary-Time Lattice Gauge Hamiltonians*.

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