

# Complex-Time Spectral Triples and the Temporal Origin of Spatial Dimension, Stationary Action, Lorentzian Geometry, and Physical Structure

A self-contained non-commutative geometric formulation of the Single Monad Model and Duality of Time Theory

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## Abstract

This article develops an operator-algebraic formulation of the Single Monad Model (SMM) and the Duality of Time Theory (DTT). The guiding hypothesis is that physical spacetime is not primitive, but is reconstructed from a layered temporal substrate: compact inner-time re-creation cycles generate discrete spectral structure, while extended outer time supplies the smooth ordering of completed events. The construction is organized as a conditional reconstruction program whose assumptions and outputs are stated explicitly.

First, the ordinary quantum phase is not postulated as a primitive scalar. Starting from a real Hilbert space with orthogonal symmetry, admissible phase units are identified with skew-adjoint square roots of  $-\text{Id}$  in the Real commutant, and a distinguished positive-time flow selects the physical complex structure by the polar factorization  $A = J|A| = |A|J$  on the dynamic sector. Second, the temporal operator algebra is formulated so that bounded algebra elements, unbounded generators, dense domains, compact-resolvent models, and non-compact local or semifinite variants are separated. Third, the dimensionality problem is treated through two complementary mechanisms: a quaternionic stability criterion selects the three-dimensional vector part  $\Im\mathbb{H} \cong \mathbb{R}^3$  under explicit associative, norm-preserving, nonabelian assumptions, while the spectral-triple realization gives infrared spectral dimension  $d_s = 3$  through Weyl growth on a three-pair inner-time lattice. Finite-regulator checks illustrate the corresponding cubic eigenvalue counting and heat-trace plateau.

The dimensional-flow layer is formulated spectrally. The stable three-pair lattice is the infrared sector of a possibly multiscale inner-time spectrum; inner time may therefore have a running spectral dimension  $d_{\text{in}}$ , while outer time remains the smooth completed ordering parameter. The effective spacetime spectral dimension is organized as  $d_{\text{eff}} = 1 + d_{\text{in}}$ , and the notation distinguishes the inner Hausdorff/counting dimension  $d_H^{\text{in}}$ , the inner spectral dimension  $d_{\text{in}}$ , and any effective gravitational or Gauss-law dimension  $d_g$ . Stationary classical action is reconstructed from one-tick  $J$ -phase kernels: kernel composition gives additive action, stationary phase gives the discrete Euler-Lagrange equations, and a local normal-form theorem explains the appearance of natural Lagrangians  $\frac{1}{2}g(v, v) + A(v) - V$ .

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A Lorentzian realization layer is then described by bounded microscopic propagation, outer-time foliation, second-moment calibration, mixing, clustering, and infrared effective-field-theory reasoning. Gauge structure is treated through ordinary non-commutative-geometric inner fluctuations and through protected inner-time eigenspace bundles whose adiabatic holonomies supply  $U(r)$  connections. A rank pattern 1, 2, 3, supplemented by quaternionic and oriented reductions, gives a conditional route to  $U(1) \times SU(2) \times SU(3)$ ; Standard-Model chirality, hypercharge normalization, Higgs breaking, and Yukawa data are left as open reconstruction problems. Stable observer records are placed downstream of phase selection and spectral dynamics; under finite-dimensional stable-record hypotheses, Born-form probabilities are recovered as trace weights on record-generated effects.

The resulting framework is not presented as a completed derivation of all known physics. It is a self-contained mathematical-physics program connecting temporal ontology, Real-commutant phase selection, temporal spectral triples, spectral dimension, stationary action, inertia, gauge holonomy, Lorentzian emergence, and stable records, while keeping conjectural links to mass hierarchies, structural constants, Yang-Mills gaps, zeta spectra, black-hole information, and the cosmological constant separate from theorem-level claims.

**Keywords:** non-commutative geometry; spectral triples; Duality of Time Theory; Single Monad Model; real Hilbert spaces; Real commutant; emergent spacetime; spectral dimension; fractal inner time; dimensional flow; adiabatic holonomy; protected eigenspace bundles; gauge symmetries; mass hierarchy; stationary action; quantum foundations; stable records; quaternionic stability; Lorentzian emergence; rotational diffusion

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# 1 Introduction

General relativity and quantum theory encode two radically successful but conceptually different descriptions of nature. In general relativity, geometry is represented by a smooth Lorentzian manifold whose metric responds to energy and momentum [1, 2]. In quantum theory, physical states, observables, and measurement probabilities are encoded by Hilbert-space amplitudes, non-commuting operators, and spectral data [3, 4]. Their tension is most acute near singularities, at the Planck scale, and in attempts to understand the origin of spacetime itself [5, 6].

Non-commutative geometry (NCG) provides one of the most precise languages for replacing point-set geometry by algebra and spectrum. In Connes' formulation, a compact Riemannian spin manifold can be recovered from a commutative spectral triple  $(\mathcal{A}, \mathcal{H}, D)$ ; when  $\mathcal{A}$  is non-commutative, the same formalism describes generalized spaces for which points are secondary or unavailable [7, 8, 9]. The Connes-Chamseddine spectral action has also shown that internal gauge structure can be encoded in finite non-commutative geometry coupled to the usual spacetime manifold [10, 11].

The SMM/DTT program inverts the usual order. Instead of beginning with spacetime and adding quantum fields, it begins with a layered temporal ontology. The Single Monad Model (SMM) asserts that multiplicity is produced by repeated re-creation from a single underlying source; the Duality of Time Theory (DTT) distinguishes compact inner-time cycles from the continuous outer-time order in which physical processes appear [12, 13]. In the spectral-triple formulation developed here, the algebra of temporal re-creation, shift, phase, and projection operators becomes the source of geometry. Spatial dimension, mass, charge, gauge structure, and observable records are treated as spectral and representation-theoretic features of time-generated operators.

The aim is to formulate this temporal reconstruction in a way that can be evaluated by standard mathematical-physics criteria. The construction therefore separates assumptions from consequences at each layer. Real Hilbert dynamics supplies the phase-selection problem; temporal crossed products and Dirac-type operators supply the spectral-triple layer; Weyl growth and heat traces supply the dimensional layer; kernel composition supplies the action layer; bounded propagation supplies the Lorentzian cone layer; protected eigenspace bundles and inner fluctuations supply gauge candidates; stable records supply the probability layer.

The first structural step concerns quantum phase. A real Hilbert space with orthogonal symmetry does not automatically carry a preferred complex structure. Compatible phase units are the skew-adjoint square roots of  $-\text{Id}$  in the Real commutant of the representation. A positive-time flow with real skew-adjoint generator  $A$  then selects  $J$  by the polar decomposition

$$A = J|A| = |A|J. \quad (1)$$

This separates the hyperbolic split-complex symbol used for inner/outer temporal duality from the circular quantum phase operator used in Hilbert-space amplitudes.

The second structural step concerns the spectral triple. The represented algebra contains bounded temporal observables, while unbounded generators are treated as affiliated operators or as parts of the Dirac operator. A model built on  $\ell^2(\mathbb{Z}) \otimes L^2(\mathbb{R})$  has a continuous outer-time spectrum, so compact resolvent is not available unless one compactifies the outer-time direction, works in a finite box, or passes to a local or semifinite spectral-triple variant. The formulation below states these alternatives explicitly.

The third structural step concerns dimension. A one-dimensional clock generator has one-dimensional Weyl growth. To obtain three extended spatial dimensions, the clock sector must organize into three

independent paired modes or an equivalent degeneracy structure. The stable infrared model used here is a three-pair temporal lattice. The statement that space is three-dimensional is therefore expressed as the spectral statement  $N_D(\Lambda) \sim C\Lambda^3$  or  $d_s(\sigma) \rightarrow 3$  in the infrared.

The fourth structural step adds a multiscale inner-time refinement. Fractal language is used only in a spectral sense: the inner-time eigenvalue-counting or heat-trace exponent may run with scale, while the observer-level outer-time coordinate remains smooth. The effective spacetime spectral dimension is written as the smooth outer-time contribution plus the running inner-sector contribution,

$$d_{\text{eff}} = 1 + d_{\text{in}}. \quad (2)$$

The notation keeps distinct the inner Hausdorff/counting dimension  $d_H^{\text{in}}$ , the inner spectral dimension  $d_{\text{in}}$ , and any effective gravitational or Gauss-law dimension  $d_g$ .

The fifth structural step concerns gauge structure. Inner fluctuations of  $D$  give the usual non-commutative-geometric one-form picture, while isolated scale-windowed eigenspace bundles of the inner-time operator carry Berry-Simon-Wilczek-Zee connections under adiabatic transport [27, 28]. In this setting, rank-one clusters give  $U(1)$  holonomy, a preserved quaternionic structure on a rank-two cluster gives  $SU(2)$ , and a preserved complex volume form on a rank-three cluster gives  $SU(3)$ . This provides a conditional gauge-holonomy mechanism, not a derivation of the full Standard Model representation content.

The sixth structural step connects temporal spectra to dynamics. Local one-tick transition kernels of the form

$$K_\varepsilon(q', q) = a_\varepsilon(q', q) \exp\left(\frac{J}{\hbar} G_\varepsilon(q, q')\right) \quad (3)$$

compose by integration, and in the semiclassical limit stationary phase selects the classical histories. Under locality, covariance, mechanical scaling, and non-degenerate quadratic stability, the leading continuum Lagrangian has the natural form

$$L(q, v) = \frac{1}{2} g_q(v, v) + A_q(v) - V(q). \quad (4)$$

The seventh structural step places probability and measurement downstream. Measurement is not used to create the quantum phase. Phase comes first, then  $J$ -linear quantum kinematics, then phase-balanced composition of systems, and only then stable record formation. Under finite-dimensional stable-record hypotheses, probabilities are trace weights  $\text{Tr}(\rho E)$  on record-generated effects. This preserves the standard non-commutative event structure and avoids turning DTT into a hidden-variable collapse mechanism.

The resulting article is a self-contained conditional reconstruction program. Some statements are theorem-level consequences of explicit assumptions; some are model-level derivations; and some are research targets. The purpose of the status table below is to make those distinctions visible.

## 1.1 Main contributions

The main contributions are the following.

1. It gives a layered reconstruction chain

real temporal dynamics  $\rightarrow$  Real-commutant phase selection  $\rightarrow$   $J$ -linear quantum kinematics  
 $\rightarrow$  temporal spectral triples  $\rightarrow$  emergent geometry and action.

(5)

2. It distinguishes the split-complex temporal unit  $j$  satisfying  $j^2 = +1$  from the quantum phase operator  $J$  satisfying  $J^2 = -\text{Id}$ .
3. It formulates a bounded temporal algebra with unbounded generators treated as affiliated or as part of the Dirac operator, thereby avoiding the common mistake of placing arbitrary unbounded operators inside the algebra of a spectral triple.
4. It supplies compact and non-compact variants of the temporal spectral triple: a compactified model with compact resolvent and a local/semifinite model suitable for outer time  $\mathbb{R}$ .
5. It explains how spectral dimension three arises from three paired inner-time modes or equivalent multiplicity growth, rather than from a single clock index.
6. It adds a fractal inner-time dimensional-flow layer:  $d_{\text{in}}$  may run with scale while outer time remains the smooth completion order, giving  $d_{\text{eff}} = 1 + d_{\text{in}}$ .
7. It distinguishes  $d_H^{\text{in}}$ ,  $d_{\text{in}}$ , and  $d_g$ , preventing the fractal refinement from being mistaken for a literal fractal observer time or an immediate change in gravitational dimensional analysis.
8. It connects the spectral triple to temporal transition kernels and stationary action, including a normal-form theorem for natural Lagrangians and a Schur-complement mechanism for effective inertia.
9. It reformulates the dimensional stability claim as a conditional algebraic stability criterion rather than an unconditional proof of the uniqueness of three-dimensional space.
10. It adds a protected-eigenspace-bundle interface for gauge emergence: adiabatic holonomy of scale-windowed inner-time clusters produces conditional  $U(1)$ ,  $SU(2)$ , and  $SU(3)$  factors under explicit rank and structure-reduction assumptions.
11. It locates measurement and Born-form probabilities in a downstream stable-record layer that preserves non-commutativity and contextuality constraints.

## 1.2 Claim status

Because the framework is ambitious, it is important to state the status of its claims. [Table 1](#) summarizes the main layers.

Table 1: Status of the main claims in the reconstruction program.

| Layer                          | Status in this article                                                                                                                                                    |
|--------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Real Hilbert substrate         | Assumed effective starting data; used to avoid assuming complex quantum phase at the outset.                                                                              |
| Real-commutant phase selection | Proved conditionally: compatible complex structures are skew-adjoint square roots of $-\text{Id}$ in the Real commutant; positive time selects $J$ on the dynamic sector. |
| Temporal spectral triple       | Constructed in compact and local/semifinite variants; domain and resolvent assumptions are stated explicitly.                                                             |

|                              |                                                                                                                                                                                                                                                                                            |
|------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Emergent spatial dimension   | Derived as spectral growth for three paired inner-time modes; the deeper reason for exactly three pairs is formulated as a stability principle.                                                                                                                                            |
| Fractal inner-time flow      | Introduced as a spectral/multiscale refinement of the inner sector; phenomenological until derived from microscopic DTT recursion. The reconstructed outer-time order remains smooth in the infrared cone and action layers.                                                               |
| Stationary action            | Reconstructed conditionally from $J$ -phase kernels and stationary phase; local kernel normal form explains natural Lagrangians.                                                                                                                                                           |
| Inertia                      | Derived within a quadratic hidden-projection model by Schur complement; relational/Machian completion remains a model-level proposal.                                                                                                                                                      |
| Gauge structure              | Inner fluctuations of $D$ yield gauge potentials in the NCG sense. A protected-cluster holonomy route gives a conditional $U(1) \times SU(2) \times SU(3)$ target from rank 1, 2, 3 bundles with additional structure reductions; exact Standard Model reconstruction remains future work. |
| Mass hierarchy               | Treated only as a spectral target: lacunary or log-periodic inner spectra can generate clustered mass scales, but observed fermion masses, Koide-type relations, and Yukawa data are not derived here.                                                                                     |
| Stable records and Born rule | Included as a downstream finite-dimensional reconstruction principle; deriving universal record dilation from microscopic DTT is open.                                                                                                                                                     |
| Open problems                | Connections to Yang-Mills, zeta spectra, black-hole information, and the cosmological constant are conjectural research bridges, not claimed solutions.                                                                                                                                    |

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## 2 Conceptual foundations of SMM/DTT

DTT distinguishes two temporal roles. Inner time labels compact, discrete, re-creative cycles. Outer time labels the continuous order in which completed configurations appear to observers. A symbolic notation often used in DTT is the hyperbolic or split-complex expression

$$t = t_i + jt_o, \quad j^2 = +1, \quad (6)$$

where  $t_i$  is associated with compact inner recurrence and  $t_o$  with extended outer flow. This notation is ontological and kinematic: it captures the two-sided temporal structure. It is not the same object as the complex phase operator  $J$  introduced later.

### 2.1 The foundational postulate

The central DTT postulate can be stated as follows.

**Principle 2.1** (Temporal re-creation). *At every completed instant of outer time, observable spatial structure is regenerated through an ordered sequence of inner-time cycles. The apparent continuity of the world is the outer-time envelope of discrete inner-time re-creation.*

This principle has three consequences.

1. **Discreteness and continuity are complementary.** The inner layer is cyclic and spectral; the outer layer is extended and continuous.
2. **Geometry is derivative.** Spatial extension is not assumed at the fundamental level; it is encoded by the pattern of inner-time recurrence.
3. **Non-commutativity is temporal.** The order of inner-time generation and outer-time projection matters. Shift, phase, and generator operators need not commute.

A minimal mathematical model begins with an inner-time sequence labelled by  $n \in \mathbb{Z}$  and a shift  $U |n\rangle = |n+1\rangle$ . A number or frequency generator  $N |n\rangle = n |n\rangle$  satisfies

$$[N, U] = U. \quad (7)$$

With a fundamental angular frequency  $\omega_0$ , the inner-time generator is  $T = \omega_0 N$  and

$$[T, U] = \omega_0 U. \quad (8)$$

This elementary relation is the seed of the non-commutative temporal algebra. It says that applying a shift and measuring the inner-time frequency are order-dependent operations.

## 2.2 Paired cycles and the emergence of spatial axes

The DTT picture associates spatial axes with pairs of inner-time cycles: two cycles stabilize one direction, four stabilize a two-dimensional sheet, six stabilize a three-dimensional volume, and a seventh role appears as outer-time continuation. In the spectral formulation this is expressed not as a pictorial numerology but as a representation-theoretic statement: stable spatial axes correspond to paired modes whose positive and negative orientations are related by phase selection.

Let

$$\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3 \quad (9)$$

label three independent paired inner modes after phase orientation has been selected. The corresponding Hilbert space is

$$\mathcal{H}_{\text{in}} = \ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^s, \quad (10)$$

where  $\mathbb{C}^s$  is a finite spinor or internal multiplicity space. The three generators

$$N_a |\mathbf{n}\rangle = n_a |\mathbf{n}\rangle, \quad U_a |\mathbf{n}\rangle = |\mathbf{n} + e_a\rangle, \quad a = 1, 2, 3, \quad (11)$$

satisfy

$$[N_a, U_b] = \delta_{ab} U_b, \quad [U_a, U_b] = 0, \quad [N_a, N_b] = 0. \quad (12)$$

The non-commutativity is local to each shift-generator pair, while the three axes commute at the coarse spatial level. This is exactly the structure needed for a spectral dimension three Weyl law.

## 2.3 Fractal inner time and smooth outer time

The phrase “fractal time” is used here only in a spectral sense. It does not mean that the observer-level outer-time coordinate is replaced by a nowhere-differentiable curve, nor that the Lorentzian

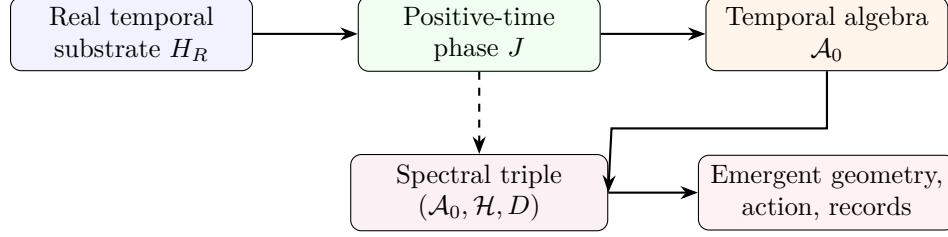


Figure 1: Layered reconstruction. The reconstruction separates real temporal dynamics, quantum phase selection, temporal operator algebra, spectral geometry, and downstream physical structures.

foliation used later is abandoned. It means that the compact inner-time spectrum may have a scale-dependent counting exponent before it settles into the stable three-pair infrared regime.

Let  $D_{\text{in}}$  be the inner-time part of the temporal Dirac operator. Define the counting and heat-trace exponents by

$$d_{\text{in}}^{\text{count}}(\Lambda) = \frac{d \log N_{D_{\text{in}}}(\Lambda)}{d \log \Lambda}, \quad d_{\text{in}}^{\text{heat}}(\sigma) = -2 \frac{d \log \text{Tr} e^{-\sigma D_{\text{in}}^2}}{d \log \sigma} \quad (13)$$

whenever the corresponding scale window is meaningful. The inner-time sector is called fractal or multiscale when these exponents are not constant over all scales, and especially when they take non-integer values over an intermediate window.

The DTT distinction between inner and outer time then gives a clean dimensional bookkeeping rule:

$$d_{\text{eff}}(E) = 1 + d_{\text{in}}(E). \quad (14)$$

The first term is the smooth completed outer-time ordering parameter. The second term is the spectral dimension generated by the inner-time modes that remain coherent at the scale  $E$ . Thus the stable infrared sector has

$$d_{\text{in}} \rightarrow 3, \quad d_{\text{eff}} \rightarrow 4, \quad (15)$$

whereas a Planckian dimensional-reduction scenario can be represented by

$$d_{\text{in}} \rightarrow 1, \quad d_{\text{eff}} \rightarrow 2. \quad (16)$$

This is the disciplined way to import recursive or fractal inner-time language into the spectral-triple construction: fractality belongs to the inner spectral measure, while outer time remains the coarse completed order used by the cone and stationary-action reconstructions.

It is also important not to identify all effective dimensions. The symbol  $d_H^{\text{in}}$  refers to the Hausdorff or counting dimension of a possible inner-time support,  $d_{\text{in}}$  to the heat-trace or eigenvalue-counting spectral dimension of the inner Dirac sector, and  $d_g$  to any effective gravitational or Gauss-law dimension used in dimensional analysis of  $G$ . A microscopic model may relate these quantities, but the spectral-triple dimensional-flow claim below uses only  $d_{\text{in}}$ .

### 3 Microdynamic route to three spatial dimensions

The spectral counting argument of [Section 8](#) shows that three paired inner modes have spectral dimension three. A deeper question remains: why should the stable sector contain three pairs rather than one, two, four, or more? The argument below separates two claims. The first is a theorem-level

conditional statement: under an explicit stable-composition interface, the only nonabelian associative normed real composition envelope is quaternionic, and its vector part is three-dimensional. The second is a broader universality claim: DTT microdynamics may flow toward that quaternionic sector through coarse-graining, isotropy, and neutral-update composition. This section states the theorem-level core and records the universality route as a target.

### 3.1 Operational assumptions

Let  $\mathfrak{G}$  denote a semigroup or group of coarse-grained inner-time update maps acting on an effective state space. The minimal operational content needed for dimensional selection is the following.

**Assumption 3.1** (Stable finite update envelope). *There is a finite-dimensional real vector space  $V$  of effective update modes with a bilinear product  $\cdot$ , a unit element 1, and a positive quadratic norm  $N(v) = \|v\|^2$  such that:*

- (i) *update composition descends to the product on  $V$ ;*
- (ii) *coarse reblocking is associative,  $(ab)c = a(bc)$ ;*
- (iii)  *$N(ab) = N(a)N(b)$  for stable updates;*
- (iv) *there are noncommuting stable unit elements;*
- (v) *nonzero stable updates are invertible, or equivalently the envelope has no zero divisors in the stable sector.*

The physical reading is direct. Associativity means that a completed multi-tick history cannot depend on arbitrary parenthesization. Multiplicativity means that the coherence magnitude of a composed update factors. Noncommutativity means that the order of elementary projections or rotations matters, which is necessary for a genuinely rotational quantum-like sector.

A more dynamical derivation of the norm starts from a coherence functional. Let  $\mathcal{N}(x) > 0$  be a stable coarse coherence magnitude on the state space and let  $U \in \mathfrak{G}$  be an admissible update. Define the one-step log-change

$$\Delta_U(x) = \log \mathcal{N}(Ux) - \log \mathcal{N}(x). \quad (17)$$

If  $\mu$  is a stationary stable ensemble and the integrability assumptions hold, then

$$\Lambda(U) = \int \Delta_U(x) d\mu(x) \quad (18)$$

satisfies the cocycle identity

$$\Lambda(UV) = \Lambda(U) + \Lambda(V), \quad (19)$$

and therefore

$$Q(U) = \exp(2\Lambda(U)) \quad (20)$$

is multiplicative. Under isotropy of the effective mode space, this multiplicative invariant is represented by a Euclidean quadratic norm after a choice of scale. Thus the multiplicative norm in Assumption 3.1 can be viewed either as a structural axiom or as the infrared form of a stationary coherence cocycle.

### 3.2 Quaternionic selection theorem

**Theorem 3.2** (Quaternionic dimensional selection). *Assume Assumption 3.1. Then the stable nonabelian associative real composition envelope is isomorphic to the quaternion algebra  $\mathbb{H}$ . Its non-scalar vector part is*

$$\Im\mathbb{H} = \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \cong \mathbb{R}^3, \quad (21)$$

*unit quaternions act on  $\Im\mathbb{H}$  by conjugation, and this action realizes the double covering*

$$SU(2) \longrightarrow SO(3). \quad (22)$$

*Consequently, within this stable-composition interface, the minimal noncommutative associative rotational sector has three spatial directions.*

*Proof.* A finite-dimensional real algebra with unit, positive multiplicative norm, and no zero divisors is a real normed division algebra. If associativity is required, Frobenius' classification [21, 22] leaves only  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$ . The real and complex cases are commutative, so the noncommuting stable-unit requirement excludes them. Hence the stable envelope is  $\mathbb{H}$ . The purely imaginary quaternions form a three-dimensional real vector space. Unit quaternions  $q$  act by  $v \mapsto qvq^{-1}$  on  $\Im\mathbb{H}$ , preserving the Euclidean norm and producing rotations. The kernel of this action is  $\{\pm 1\}$ , giving the usual double cover  $SU(2) \simeq Sp(1) \rightarrow SO(3)$ .  $\square$

*Remark 3.3* (What is actually derived). The theorem does not say that every conceivable temporal model must be quaternionic. It says that once the stable-composition interface is imposed—finite dimensionality, associativity, multiplicative positive norm, and genuine noncommutativity—the quaternionic envelope and its three-dimensional vector part are forced. This is considerably stronger than merely postulating three axes, but it remains conditional on the interface.

### 3.3 H-space universality route

There is a complementary topological route. Let  $K = \text{Ker } Q$  be the neutral update sector. Under boundedness and closure assumptions, the effective representation of  $K$  closes to a compact Lie group. If the orbit of a reference direction is a direction manifold  $X \simeq K/H$ , and if the neutral-update composition induces a multiplication on  $X$  up to homotopy, then  $X$  is an  $H$ -space. The Hopf-invariant-one theorem [23] restricts spheres admitting such multiplication to

$$S^0, \quad S^1, \quad S^3, \quad S^7. \quad (23)$$

These correspond to the normed division algebra pattern  $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ . If associative composition is retained, the octonionic  $S^7$  case is excluded by non-associativity. Requiring a nonabelian rotational sector then leaves the quaternionic case as the first viable associative noncommutative option. This route is not used as the main proof, but it is valuable because it ties the dimensional selection problem to a standard topological rigidity theorem rather than to model-specific numerology.

### 3.4 Consequences for the spectral-triple construction

The quaternionic theorem supplies the structural input that the spectral model of Sections 6 and 8 needs. The scalar part of  $\mathbb{H}$  represents the identity or outer-time calibration component, while the

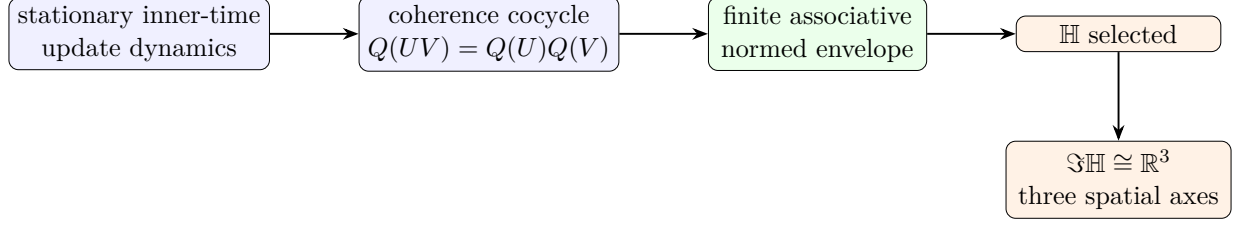


Figure 2: Microdynamic-to-quaternionic route. The key selection theorem is conditional: the stable finite associative normed nonabelian envelope is quaternionic, and its vector part is three-dimensional.

vector part  $\Im\mathbb{H}$  supplies the three stable spatial projection directions. After positive-time phase selection, these three directions are represented by three paired inner modes

$$(N_1, U_1), \quad (N_2, U_2), \quad (N_3, U_3), \quad (24)$$

whose crossed-product algebra is associative and noncommutative. The theorem therefore supports the use of a  $\mathbb{Z}^3$  inner cycle-pair lattice as the stable infrared sector rather than treating it as an arbitrary modeling choice.

## 4 Real Hilbert dynamics and phase selection

A major weakness of many foundational approaches is that the complex Hilbert space is simply assumed. DTT can do better. If the primitive temporal substrate is real, the existence of a quantum phase unit must be explained. This section supplies the needed structure.

### 4.1 Real commutants and admissible phase units

Let  $H_R$  be a real Hilbert space and let  $U : G \rightarrow O(H_R)$  be an orthogonal representation of a symmetry group  $G$ . Its complexification is

$$H_C = H_R \otimes_R \mathbb{C}, \quad (25)$$

with canonical conjugation  $\Gamma(x + iy) = x - iy$ . Let  $V(g)$  denote the complexified unitary representation.

**Definition 4.1** (Real commutant). *The Real commutant of  $U$  is*

$$\mathcal{A}_R(U) = \{T \in \mathcal{B}(H_C) : TV(g) = V(g)T \ \forall g \in G, \ T\Gamma = \Gamma T\}. \quad (26)$$

*Equivalently, it is the complexification of the real commutant*

$$\text{Comm}_R(U) = \{S \in \mathcal{B}(H_R) : SU(g) = U(g)S \ \forall g \in G\}. \quad (27)$$

**Definition 4.2** (Compatible phase unit). *An orthogonal complex structure compatible with  $U$  is a real-linear bounded operator  $J : H_R \rightarrow H_R$  satisfying*

$$J^* = -J, \quad J^2 = -\text{Id}, \quad JU(g) = U(g)J \quad \forall g \in G. \quad (28)$$

*The set of all such structures is denoted  $\mathcal{J}(U)$ .*

**Theorem 4.3** (Real-commutant phase classification). *Complexification gives an isometric real  $*$ -isomorphism*

$$\text{Comm}_R(U) \longrightarrow \mathcal{A}_R(U), \quad S \mapsto S_C. \quad (29)$$

*Consequently, compatible phase units are precisely the skew-adjoint square roots of  $-\text{Id}$  in the Real commutant.*

*Proof.* If  $S$  is real-linear, bounded, and commutes with  $U(g)$ , then its complexification  $S_C$  is complex-linear, commutes with  $V(g)$ , and commutes with  $\Gamma$ . Conversely, if  $T \in \mathcal{A}_R(U)$ , then  $T$  preserves  $\text{Fix}(\Gamma) = H_R$ , and its restriction is a real-linear operator commuting with  $U$ . The algebraic relations  $J^* = -J$  and  $J^2 = -\text{Id}$  are preserved by complexification and restriction.  $\square$

Thus the imaginary unit is not a primitive scalar at the deepest layer. It is an operator selected inside a real symmetry algebra. Only after  $J$  is selected do we define complex scalar multiplication by

$$(a + ib)x = ax + bJx, \quad a, b \in \mathbb{R}, \quad (30)$$

and the complex inner product

$$\langle x, y \rangle_J = (x, y)_R - i(Jx, y)_R. \quad (31)$$

## 4.2 Positive-time polar selection

Classification is not selection. A physical theory needs to choose the phase orientation associated with positive time.

Let  $U(t) = e^{tA}$  be a strongly continuous orthogonal flow on  $H_R$ . The generator  $A$  is real skew-adjoint. Define

$$H_{\text{dyn}} = (\text{Ker } A)^\perp. \quad (32)$$

**Theorem 4.4** (Positive-time polar selection). *On  $H_{\text{dyn}}$ , the polar decomposition of the real skew-adjoint generator  $A$  yields a unique orthogonal complex structure  $J_A$  satisfying*

$$A = J_A |A| = |A| J_A, \quad |A| = (A^* A)^{1/2} \geq 0. \quad (33)$$

*If  $\tilde{J}$  is another orthogonal complex structure and  $A = \tilde{J}\tilde{B} = \tilde{B}\tilde{J}$  with  $\tilde{B} \geq 0$ , then  $\tilde{B} = |A|$  and  $\tilde{J} = J_A$  on  $H_{\text{dyn}}$ .*

*Proof.* The closed skew-adjoint operator  $A$  has polar decomposition  $A = V|A|$ . On  $H_{\text{dyn}}$  the partial isometry  $V$  is unitary. Skew-adjointness gives  $V^* = -V$  and  $V^2 = -\text{Id}$  on the support of  $|A|$ . Set  $J_A = V$ . Uniqueness follows from uniqueness of polar decomposition on the support of  $|A|$ .  $\square$

Zero modes are not failures of the theory. They are precisely those directions on which the time flow has no orientation. They may represent constraints, gauge modes, static parameters, or degrees of freedom requiring extra data.

### 4.3 Discrete ticks and phase aliasing

DTT emphasizes discrete inner-time ticks. A one-tick orthogonal operator  $W_\varepsilon$  determines a generator only after a logarithm branch is selected.

**Proposition 4.5** (Tick logarithm). *Let  $W_\varepsilon \in O(H_R)$  be a one-tick operator. Suppose the spectrum of its complexification lies in an arc of the unit circle avoiding  $-1$ . Then there is a unique real skew-adjoint generator  $A$  with spectrum in the corresponding branch interval such that*

$$W_\varepsilon = e^{\varepsilon A}. \quad (34)$$

The polar-selection theorem then selects  $J_A$  on  $(\text{Ker } A)^\perp$ .

If the spectrum wraps around the unit circle, the logarithm is not unique. Energies separated by approximately

$$E_{\text{alias}} \sim \frac{\pi \hbar}{\varepsilon} \quad (35)$$

produce the same one-tick phase. If  $\varepsilon$  is physical rather than merely a regulator, this predicts a possible high-energy aliasing scale.

### 4.4 The distinction between $j$ and $J$

The split-complex unit  $j$  and the phase operator  $J$  play different roles:

$$j^2 = +1, \quad J^2 = -\text{Id}. \quad (36)$$

The former encodes the hyperbolic inner/outer temporal split; the latter encodes circular quantum phase in a selected Hilbert sector. This distinction prevents a common confusion between complex-time notation and the ordinary quantum phase.

## 5 Temporal operator algebras

A spectral triple requires a represented algebra of bounded operators and an unbounded Dirac operator. The temporal framework uses unbounded generators such as  $N$ ,  $T$ , and  $H$ , so the algebra must be formulated with care.

### 5.1 The inner crossed-product algebra

For one inner clock, let  $\mathcal{H}_{\text{in}} = \ell^2(\mathbb{Z})$ , let  $N|n\rangle = n|n\rangle$ , and let  $U|n\rangle = |n+1\rangle$ . The shift is bounded and unitary; the number operator is unbounded. A natural bounded algebra is the crossed product

$$\mathcal{A}_{\text{in}} = c_0(\mathbb{Z}) \rtimes \mathbb{Z}, \quad (37)$$

represented on  $\ell^2(\mathbb{Z})$  by diagonal multiplication operators  $f(N)$  with  $f \in c_0(\mathbb{Z})$  and shifts  $U^k$ . A smooth dense subalgebra is

$$\mathcal{A}_{\text{in}}^\infty = \left\{ \sum_{|k| \leq K} f_k(N) U^k : K < \infty, f_k \in \mathcal{S}(\mathbb{Z}) \right\}, \quad (38)$$

where  $\mathcal{S}(\mathbb{Z})$  denotes rapidly decreasing sequences. The commutator with  $N$  is bounded on this algebra:

$$[N, f(N)U^k] = kf(N)U^k. \quad (39)$$

For three paired inner modes one uses

$$\mathcal{H}_{\text{in}} = \ell^2(\mathbb{Z}^3), \quad \mathcal{A}_{\text{in}}^\infty = \mathcal{S}(\mathbb{Z}^3) \rtimes_{\text{alg}} \mathbb{Z}^3, \quad (40)$$

with

$$[N_a, f(\mathbf{N})U^{\mathbf{k}}] = k_a f(\mathbf{N})U^{\mathbf{k}}. \quad (41)$$

## 5.2 Outer time: compact and non-compact models

If outer time is represented by  $L^2(\mathbb{R})$  and  $H_o = -i\partial_t$ , the spectrum is continuous and a standard compact-resolvent spectral triple is not obtained. There are three legitimate options.

1. **Compactified outer-time model.** Replace  $\mathbb{R}$  by a circle  $S_R^1$  of large period  $2\pi R$ . Then  $H_o = -i\partial_t$  has discrete spectrum and compact-resolvent constructions are available.
2. **Local compactness.** Work with  $L^2(\mathbb{R})$  and require  $a(1 + D^2)^{-1/2}$  to be compact for  $a$  in a suitable non-unital algebra, rather than demanding compact resolvent of  $D$  itself.
3. **Semifinite spectral triples.** Use a trace on a von Neumann algebra adapted to the continuous outer-time direction, as in semifinite NCG.

The compactified model is sufficient for a finite-volume or infrared-regulated construction. The non-compact variants are more physical but technically more demanding.

## 5.3 Temporal algebra with phase and finite internal structure

The full working algebra is a tensor product

$$\mathcal{A}_0 = \mathcal{A}_{\text{in}}^\infty \otimes \mathcal{A}_{\text{out}}^\infty \otimes \mathcal{A}_F, \quad (42)$$

where  $\mathcal{A}_{\text{out}}^\infty$  is either  $C^\infty(S_R^1)$  or  $C_c^\infty(\mathbb{R})$  and  $\mathcal{A}_F$  is a finite-dimensional internal algebra. In the conservative construction,  $\mathcal{A}_F$  is left unspecified and is interpreted as an internal temporal multiplicity algebra. For comparison with Standard-Model-oriented non-commutative geometry, one may compare with the Connes-Chamseddine finite algebra

$$\mathcal{A}_F \simeq \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad (43)$$

but deriving this algebra from temporal principles remains a future task.

Phase-modified projections are represented by diagonal unitaries of the form

$$V_\theta = \sum_{\mathbf{n}} e^{i\theta(\mathbf{n})} |\mathbf{n}\rangle \langle \mathbf{n}|, \quad (44)$$

when the phase function is bounded. These implement  $U(1)$ -type temporal phase rotations. More complicated finite matrix factors can encode non-Abelian internal symmetries.

## 6 Complex-time spectral triples

The temporal spectral triple is constructed in a form suitable for mathematical inspection.

### 6.1 Compactified model

Let

$$\mathcal{H} = \ell^2(\mathbb{Z}^3) \otimes L^2(S_R^1) \otimes \mathbb{C}^r \otimes \mathbb{C}^m, \quad (45)$$

where  $\mathbb{C}^r$  carries Clifford matrices and  $\mathbb{C}^m$  carries finite internal degrees of freedom. Let  $N_a$  be the three inner generators and let  $P = -i\partial_t$  on  $S_R^1$ . Choose Hermitian Clifford matrices  $\Gamma^1, \Gamma^2, \Gamma^3, \Gamma^0$  satisfying

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu} I \quad (46)$$

for the Euclidean compact spectral model. Lorentzian signature can be treated by Krein-space refinements, but ordinary compact spectral triples are most naturally Euclidean.

Define

$$D_0 = \sum_{a=1}^3 \Gamma^a \omega_a N_a + \Gamma^0 P + D_F, \quad (47)$$

where  $D_F$  is a self-adjoint finite matrix. A bounded hopping term may also be added,

$$B_U = \sum_{a=1}^3 \left( \Gamma^{a,+} U_a + \Gamma^{a,-} U_a^* \right), \quad (48)$$

provided it is self-adjoint. Since  $B_U$  is bounded,  $D = D_0 + B_U$  has the same essential self-adjointness and compact-resolvent properties as  $D_0$ .

**Proposition 6.1** (Temporal compact spectral triple). *Let  $\mathcal{A}_0 = \mathcal{S}(\mathbb{Z}^3) \rtimes_{\text{alg}} \mathbb{Z}^3 \otimes C^\infty(S_R^1) \otimes \mathcal{A}_F$  act on  $\mathcal{H}$  as above. Then  $(\mathcal{A}_0, \mathcal{H}, D)$  is a spectral triple for the compactified model:  $D$  is essentially self-adjoint on the natural finite-support smooth domain,  $(1 + D^2)^{-1/2}$  is compact, and  $[D, a]$  is bounded for all  $a \in \mathcal{A}_0$ .*

*Proof sketch.* The operator  $\sum_a \Gamma^a \omega_a N_a + \Gamma^0 P + D_F$  is a sum of commuting self-adjoint operators on the tensor-product smooth core, with finite matrix coefficients. It is essentially self-adjoint by standard results for sums of commuting self-adjoint operators on a common core. The added hopping term is bounded and self-adjoint, so Kato-Rellich preserves essential self-adjointness [19, 20]. Compactness follows because the joint eigenvalues of  $(\mathbf{N}, P)$  escape to infinity and have finite multiplicity in the compactified model. For  $a = f(\mathbf{N}) U^{\mathbf{k}} \otimes g(t) \otimes b_F$ , the commutator with the  $N_a$  terms is multiplication by  $k_a$  times a bounded shift term, the commutator with  $P$  is multiplication by  $-ig'(t)$ , and the finite part is bounded. Finite sums give the result.  $\square$

*Remark 6.2.* The one-index model  $\ell^2(\mathbb{Z}) \otimes L^2(\mathbb{R})$  is useful for intuition, but it is not a compact spectral triple without modification. The construction used here therefore employs either three inner modes and compact outer time or a non-compact spectral-triple variant.

## 6.2 Non-compact outer-time variant

For  $\mathcal{H} = \ell^2(\mathbb{Z}^3) \otimes L^2(\mathbb{R}) \otimes \mathbb{C}^r \otimes \mathbb{C}^m$  and  $\mathcal{A}_{\text{out}}^\infty = C_c^\infty(\mathbb{R})$ , the same formula for  $D$  defines a non-compact spectral triple in the local sense:

$$a(1 + D^2)^{-1/2} \in \mathcal{K}(\mathcal{H}), \quad a \in \mathcal{A}_0, \quad (49)$$

under standard elliptic/local-compactness assumptions. If one wants a trace per unit outer time, a semifinite model can be formulated by replacing the ordinary compactness condition by compactness relative to a semifinite trace.

## 6.3 Spectral distance

Given a spectral triple, Connes' distance between states  $\varphi$  and  $\psi$  of the algebra is

$$d_D(\varphi, \psi) = \sup \{ |\varphi(a) - \psi(a)| : a = a^*, \|[D, a]\| \leq 1 \}. \quad (50)$$

In DTT, states localized around different inner-time configurations have distance determined by how sharply  $D$  changes under transitions between those configurations. Thus metric structure is not imposed on a pre-existing manifold; it is recovered from the commutator seminorm of temporal observables.

## 6.4 Inner fluctuations and gauge fields

In NCG, gauge potentials arise from inner fluctuations of the Dirac operator:

$$D \mapsto D_A = D + A + JAJ^{-1}, \quad A = \sum_i a_i [D, b_i]. \quad (51)$$

Here  $J$  may denote the real-structure operator in NCG notation; to avoid conflict with the phase-selection operator, one can denote the NCG real structure by  $\mathcal{J}_{\text{NCG}}$ . The key point is that one-forms  $A$  are built from commutators of  $D$  with algebra elements. In the temporal interpretation, gauge potentials are fluctuations of the spectral relation between inner-time cycles and outer-time propagation.

The curvature is

$$F_A = dA + A^2, \quad (52)$$

with  $da = [D, a]$  in the universal differential calculus modulo junk forms. The spectral action

$$S_{\text{spec}}(D_A) = \text{Tr } f(D_A/\Lambda) \quad (53)$$

then produces geometric and field-theoretic terms in the large- $\Lambda$  expansion when the heat-kernel machinery applies.

## 7 Cone-first Lorentzian geometry and gravitational universality

The compact spectral-triple model is Euclidean in the technical NCG sense. DTT, however, is physically Lorentzian: outer-time ordering and bounded propagation should generate a causal cone. The appropriate way to include this layer is not to claim that the Euclidean compact triple already is a Lorentzian spacetime. Instead, one separates two layers:

1. the Euclidean or semifinite spectral triple supplies the algebraic/spectral reconstruction of temporal geometry;
2. a coarse-grained realization model supplies the Lorentzian causal structure through bounded propagation, foliation, and increment statistics.

This section formulates the second layer as a theorem-target chain with explicitly stated assumptions.

## 7.1 Bounded propagation and the macroscopic cone

Let  $M_{\text{phys}}$  be the emergent quotient manifold obtained after coarse-graining the microscopic re-creation process. Let  $T : M_{\text{phys}} \rightarrow \mathbb{R}$  denote the outer-time function and let  $(T, X^1, X^2, X^3)$  be a foliation-adapted chart. Assume that one microscopic re-creation step transports any marked excitation by at most  $m_0$  intrinsic units, and let  $\ell_{\text{rec}}$  be the re-creation length scale and  $\tau_{\text{out}}$  the outer-time tick. The associated speed scale is

$$c = \frac{\ell_{\text{rec}}}{\tau_{\text{out}}} \quad (54)$$

after normalization.

**Assumption 7.1** (Bounded propagation). *For coarse-graining scale  $k$  in the continuum regime, the one-step increment satisfies, with tails vanishing in the limit,*

$$\|\Delta X^{(k)}\| \leq c \Delta T^{(k)} + o(\ell_{\text{rec}}). \quad (55)$$

**Proposition 7.2** (Cone from bounded propagation). *Under Assumption 7.1, the support of the local increment law at  $x \in M_{\text{phys}}$  lies inside the closed cone*

$$C_x = \{v \in T_x M_{\text{phys}} : \|v_{\perp}\| \leq c |v_T|\} \quad (56)$$

*up to vanishing continuum-limit tails. The pair  $(C_x, T)$  determines a time-oriented causal cone field.*

*Proof sketch.* Push forward the microscopic displacement bound through the coarse-graining map and the foliation-adapted chart. The spatial part is bounded by the maximal displacement per outer-time increment, while the sign of  $\Delta T$  supplies the time orientation. Stability of the coarse-graining limit gives a cone field in the tangent bundle.  $\square$

## 7.2 Cone plus foliation gives a Lorentzian conformal class

A cone field does not by itself fix a metric scale, but it fixes the conformal class of Lorentzian metrics compatible with that cone. In local coordinates, the standard representative is

$$g_0 = -c^2 dT^2 + \sum_{i=1}^3 dX^i dX^i. \quad (57)$$

More generally, anisotropic local second moments and coordinate changes give a position-dependent representative.

Let  $P_x$  be the local increment law at  $x$  and define the second moment tensor

$$G^{\mu\nu}(x) = \int \Delta X^\mu \Delta X^\nu dP_x(\Delta X). \quad (58)$$

Under the usual first-order transformation law for increments,  $G^{\mu\nu}$  transforms tensorially. It is a calibration tensor: it fixes local scales and anisotropies inside the conformal structure, but Lorentzian signature comes from the cone and foliation, not from a positive second-moment matrix alone.

**Theorem 7.3** (Cone-calibrated Lorentzian metric). *Assume bounded propagation, a smooth outer-time foliation, regular local increment laws, and a nondegenerate calibration tensor. Then the macroscopic cone field determines a Lorentzian conformal class  $[g]$ , and the second-moment calibration selects a representative  $g_{\mu\nu}$  up to the remaining normalization convention.*

*Proof sketch.* The cone field selects null directions and hence a Lorentzian conformal class. The outer-time foliation chooses a time orientation. The second-moment tensor supplies local normalization data for the spatial and temporal scales, fixing a representative once a convention for  $c$  and volume density is chosen.  $\square$

### 7.3 Scale-dependent completion without fractalizing outer time

If the inner-time spectrum has a multiscale or fractal regime, the completion data entering the cone construction may acquire energy dependence. The appropriate replacement is not a fractal observer time, but a family of effective completion calibrations

$$\Theta = \Theta(E), \quad \rho = \rho(E), \quad \kappa = \kappa(E), \quad (59)$$

with finite infrared limits. The smooth DTT lift then carries a scale-indexed representative

$$g_E = -\kappa(E)^2 \vartheta \otimes \vartheta + h_E, \quad c(E) = \kappa(E)\rho(E), \quad (60)$$

where  $h_E$  denotes the spatial quadratic form reconstructed after coarse graining at scale  $E$ .

The low-energy Lorentzian theorem uses the limits

$$\Theta_0 = \lim_{E \rightarrow 0} \Theta(E), \quad \rho_0 = \lim_{E \rightarrow 0} \rho(E), \quad \kappa_0 = \lim_{E \rightarrow 0} \kappa(E). \quad (61)$$

Possible ultraviolet dispersion, gravitational-wave delays, or entropy-scaling corrections belong to the running of these completion data. The observer-level outer-time foliation remains smooth in the regime where the cone theorem is applied.

### 7.4 Isotropization, locality, and Einstein-Hilbert dominance

The emergence of a Lorentzian metric is not enough. A viable continuum theory must also explain why Lorentz-violating anisotropies are suppressed, why the effective action is local, and why the Einstein-Hilbert term dominates. These are best phrased as universality assumptions and consequences.

**Assumption 7.4** (Mixing and anisotropy contraction). *The inner-time update process is stationary and mixing, with a spectral gap or sufficiently strong mixing coefficients, so that block variables obey a functional central-limit theorem and anisotropy operators are irrelevant under coarse-graining.*

**Assumption 7.5** (Clustering). *Connected correlations of bounded local observables decay exponentially or with an integrable power law on outer-time slices.*

**Theorem 7.6** (Lorentzian and gravitational universality chain). *For a microscopic dual-time realization satisfying bounded propagation, coarse-graining regularity, mixing/isotropization, clustering, and emergent diffeomorphism invariance, the infrared theory has:*

- (i) *a Lorentzian causal structure from the propagation cone;*
- (ii) *local Lorentz covariance of correlators up to  $\ell_{\text{rec}}/L$  corrections;*
- (iii) *a quasi-local effective action with a derivative expansion;*
- (iv) *an infrared gravitational action of the form*

$$S_{\text{grav}}^{\text{IR}} = \frac{1}{16\pi G_{\text{eff}}} \int d^4x \sqrt{-g} (R - 2\Lambda_{\text{eff}}) + \sum_{n \geq 2} \ell_{\text{rec}}^{2n-2} \int d^4x \sqrt{-g} \mathcal{O}_{2n}. \quad (62)$$

*The constants  $G_{\text{eff}}$ ,  $\Lambda_{\text{eff}}$ , and higher-curvature coefficients are model-dependent and are not computed by the abstract theorem.*

*Proof sketch.* Bounded propagation gives the cone and hence  $[g]$ . Mixing and an invariance principle suppress preferred microscopic directions in the block limit, yielding local Lorentz covariance. Clustering gives factorization at large separation and permits a cumulant/cluster expansion, producing a quasi-local effective action. Once locality, a derivative expansion, and diffeomorphism invariance are present, standard gravitational EFT [26, 25] implies Einstein-Hilbert dominance in the infrared, with higher-curvature terms suppressed by powers of the microscopic scale.  $\square$

*Remark 7.7* (How this relates to the spectral triple). The spectral triple describes the algebraic and spectral origin of geometry; the cone-first construction describes the Lorentzian signature and infrared gravitational dynamics of a realization model. The two are compatible but logically distinct. This separation prevents the common error of treating a compact Euclidean spectral triple as if it had already solved the Lorentzian problem.

## 8 Spectral dimension from inner-time growth

A spectral triple extracts dimension from eigenvalue growth, heat traces, and zeta functions. For  $D$  with compact resolvent, define

$$N_D(\Lambda) = \#\{\lambda \in \text{Spec}(|D|) : \lambda \leq \Lambda\}, \quad (63)$$

counting multiplicity. If

$$N_D(\Lambda) \sim C\Lambda^{d_s}, \quad (64)$$

then  $d_s$  is the spectral dimension. Equivalently,

$$d_s = -2 \lim_{\sigma \rightarrow 0} \frac{d \log \text{Tr} e^{-\sigma D^2}}{d \log \sigma} \quad (65)$$

when the limit exists. In finite regulators one replaces the limit by a scale window:  $d_s(\sigma)$  is meaningful only between the ultraviolet lattice cutoff and the infrared box size.

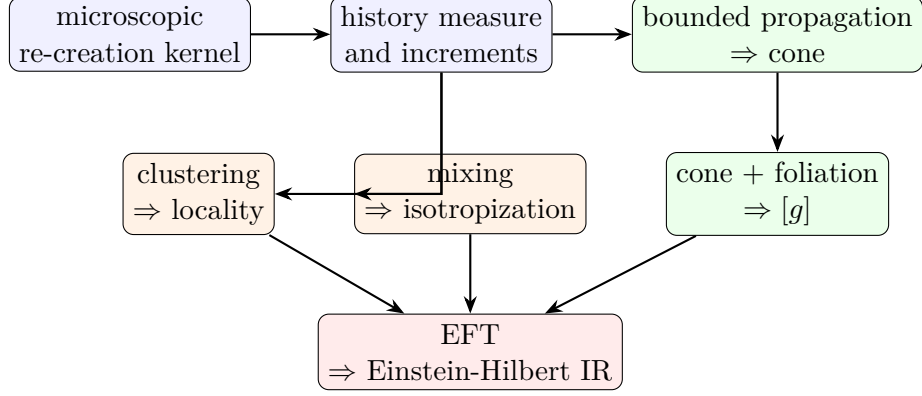


Figure 3: Cone-first Lorentzian universality chain. The Einstein-Hilbert term appears only after bounded propagation, isotropization, clustering, and diffeomorphism-invariant coarse graining are in place.

### 8.1 The one-clock obstruction

If  $D \sim \omega N$  on  $\ell^2(\mathbb{Z})$ , then  $N_D(\Lambda) \sim C\Lambda$ , so the spectral dimension is one. Therefore a single clock index cannot by itself justify three spatial dimensions. Three-dimensionality requires either three independent temporal pairs or a degeneracy law with multiplicity  $g(k) \sim k^2$ . This observation is crucial: without it, the phrase “three dimensions emerge from time” remains only pictorial.

**Proposition 8.1** (One-clock Weyl law). *Let  $D_1 = \omega N$  on  $\ell^2(\mathbb{Z})$ , with  $\omega > 0$  and  $N|n\rangle = n|n\rangle$ . Then*

$$N_{D_1}(\Lambda) = 2 \left\lfloor \frac{\Lambda}{\omega} \right\rfloor + 1 = O(\Lambda), \quad (66)$$

so the spectral dimension is  $d_s = 1$ .

### 8.2 Three paired modes and anisotropic Weyl growth

For the three-pair model, take

$$D_{\text{in}} = \sum_{a=1}^3 \Gamma^a \omega_a N_a, \quad (67)$$

where  $\omega_a > 0$ ,  $N_a|n\rangle = n_a|n\rangle$ , and the Clifford matrices satisfy  $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}I$ . Then

$$D_{\text{in}}^2 = \sum_{a=1}^3 \omega_a^2 N_a^2 \quad (68)$$

up to finite spin and internal multiplicities.

**Theorem 8.2** (Three-pair Weyl law). *Let  $D_{\text{in}}$  be the three-pair temporal Dirac operator on  $\ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^r \otimes \mathbb{C}^m$  with positive weights  $\omega_1, \omega_2, \omega_3$ . Then*

$$N_D(\Lambda) \sim rm \frac{4\pi}{3\omega_1\omega_2\omega_3} \Lambda^3, \quad \Lambda \rightarrow \infty, \quad (69)$$

where  $r$  and  $m$  are the finite Clifford and internal multiplicities. Hence  $d_s = 3$ .

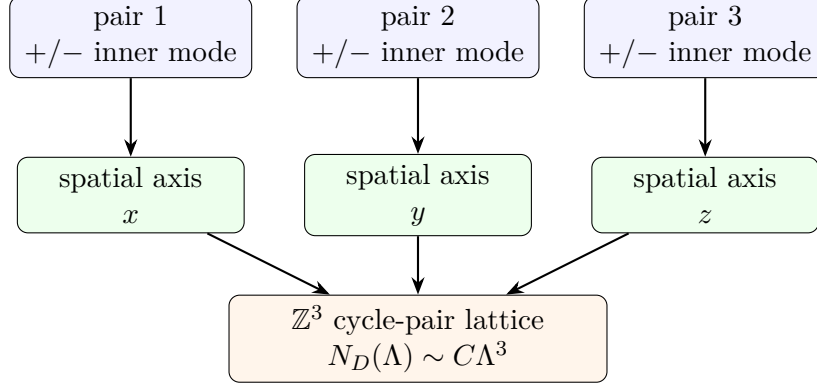


Figure 4: Spectral refinement of the DTT dimensional claim. Three paired inner-time modes generate an effective  $\mathbb{Z}^3$  spectral lattice with cubic Weyl growth, hence spectral dimension three.

*Proof.* The eigenvalues of  $|D_{\text{in}}|$  are, up to finite multiplicity,

$$\Omega(\mathbf{n}) = \left( \sum_{a=1}^3 \omega_a^2 n_a^2 \right)^{1/2}. \quad (70)$$

Thus  $N_D(\Lambda)$  counts lattice points in the ellipsoid

$$\sum_{a=1}^3 \omega_a^2 n_a^2 \leq \Lambda^2. \quad (71)$$

Standard lattice-point asymptotics give the Euclidean volume of this ellipsoid plus lower-order boundary terms. The volume is  $(4\pi/3)\Lambda^3/(\omega_1\omega_2\omega_3)$ . Finite matrix multiplicities multiply the leading coefficient but not the exponent.  $\square$

This is the sharpened spectral form of the DTT statement that six inner-time cycles generate three spatial dimensions: each stable spatial axis corresponds to a paired mode, and three paired modes yield cubic Weyl growth.

### 8.3 Finite-regulator numerical check

A finite regulator is not a proof of the Weyl law, but it is useful for a reader because it shows what the spectral claim looks like in an explicit computable model. Let

$$\mathbb{Z}_M^3 = \{-M, \dots, M\}^3 \quad (72)$$

and take the regulated eigenvalues  $\Omega(\mathbf{n}) = |\mathbf{n}|$  for  $\mathbf{n} \in \mathbb{Z}_M^3$ . The counting function is

$$N_M(\Lambda) = \#\{\mathbf{n} \in \mathbb{Z}_M^3 : |\mathbf{n}| \leq \Lambda\}. \quad (73)$$

For  $1 \ll \Lambda \ll M$ , one expects a cubic window. Figure 5 shows this directly. Figure 6 shows the corresponding heat-trace dimension

$$d_s(\sigma) = -2 \frac{d \log H_M(\sigma)}{d \log \sigma}, \quad H_M(\sigma) = \sum_{\mathbf{n} \in \mathbb{Z}_M^3} e^{-\sigma |\mathbf{n}|^2}. \quad (74)$$

The plateau near 3 is the finite-regulator signature of the three-pair sector. Deviations at very small  $\sigma$  are cutoff effects; deviations at large  $\sigma$  reflect the zero mode and finite volume.

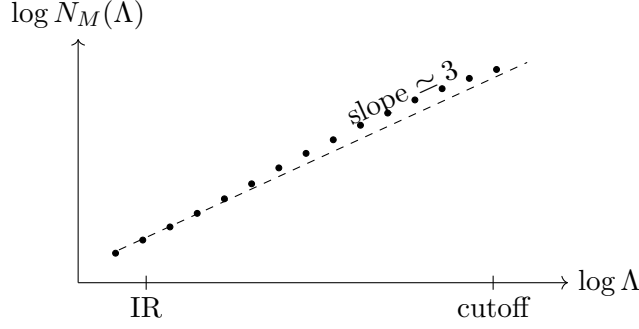


Figure 5: Finite-regulator eigenvalue-counting check for a  $\mathbb{Z}^3$  temporal cycle-pair lattice. The fitted log-log slope in the scaling window is close to three, as predicted by [Theorem 8.2](#).

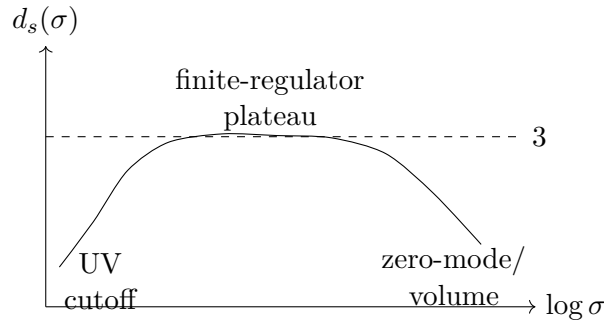


Figure 6: Heat-trace spectral dimension for the same finite regulator. The intermediate-scale plateau near  $d_s = 3$  is the numerical signature of the stable three-pair temporal sector.

#### 8.4 Fractal inner-time spectra and dimensional flow

A more general temporal spectrum may exhibit scale-dependent dimension. In this formulation, that is the precise mathematical meaning of a fractal inner-time regime. The stable three-pair lattice is the infrared fixed sector, but the ultraviolet spectral measure may have a different scaling exponent.

The dimensional dictionary is as follows. A self-similar inner support may have a Hausdorff/counting exponent  $d_H^{\text{in}}$ ; the Dirac or diffusion data have a spectral exponent  $d_{\text{in}}$ ; and gravitational dimensional analysis, if introduced, uses a generally distinct effective exponent  $d_g$ . The flow studied in this section is only the spectral one. Thus  $d_{\text{eff}} = 1 + d_{\text{in}}$  is bookkeeping for observed diffusion/eigenvalue dimension, not a statement that  $d_g$  or the topological dimension must run in the same way.

Suppose first that the effective inner dispersion is

$$\Omega(\mathbf{n}) \sim C |\mathbf{n}|^\alpha \quad (75)$$

at high frequency. Then

$$N_{D_{\text{in}}}(\Lambda) \sim C' \Lambda^{3/\alpha}, \quad (76)$$

and the inner spectral dimension becomes

$$d_{\text{in}} = \frac{3}{\alpha}. \quad (77)$$

Logarithmic corrections or scale-dependent exponents  $\alpha = \alpha(\Lambda)$  produce slow running. More generally, one may define the running inner dimension by either

$$d_{\text{in}}(\Lambda) = \frac{d \log N_{D_{\text{in}}}(\Lambda)}{d \log \Lambda} \quad (78)$$

or

$$d_{\text{in}}(\sigma) = -2 \frac{d \log \text{Tr} e^{-\sigma D_{\text{in}}^2}}{d \log \sigma}. \quad (79)$$

The two definitions agree on ordinary power-law scaling windows and differ only by regulator and crossover effects in finite models.

The full DTT bookkeeping separates the smooth completed ordering time from the inner spectral sector:

$$d_{\text{eff}}(E) = 1 + d_{\text{in}}(E). \quad (80)$$

Consequently the standard macroscopic value is recovered from

$$d_{\text{in}}(E) \rightarrow 3, \quad d_{\text{eff}}(E) \rightarrow 4 \quad (E \ll E_*), \quad (81)$$

while a dimensional-reduction regime of the type often discussed in quantum gravity corresponds here to

$$d_{\text{in}}(E) \rightarrow 1, \quad d_{\text{eff}}(E) \rightarrow 2 \quad (E \gg E_*). \quad (82)$$

A convenient phenomenological interpolation is

$$d_{\text{eff}}(E) = 2 + \frac{2}{1 + (E/E_*)^p}, \quad d_{\text{in}}(E) = 1 + \frac{2}{1 + (E/E_*)^p}, \quad p > 0, \quad (83)$$

although the preferred form should ultimately be derived from the microscopic inner-time spectrum rather than chosen by hand. Equivalently, one may write  $d_{\text{eff}}(E) = 2 + 2C(E)$  with a bounded coherence function  $C(E) = (1 + (E/E_*)^p)^{-1}$ . This makes explicit that the interpolation is a phenomenological closure: the real task is to compute  $C(E)$  from a temporal transfer operator, heat kernel, or recursive spectral decimation rule.

The ultraviolet endpoint  $d_{\text{in}} \rightarrow 1$  should likewise be read structurally, not numerologically. It represents the regime in which only one paired inner-time direction remains coherently projected, while the other two are suppressed, randomized, or anomalously dispersed. Other UV endpoints would be possible in other microscopic models; 1 is the natural endpoint for a single surviving coherent pair.

A useful operator-level way to phrase the same idea is to introduce scale-windowed inner spectra. Let  $\chi_E$  be a smooth spectral cutoff or window and let

$$D_{\text{in}}^{(E)} = \chi_E(D_{\text{in}}) D_{\text{in}} \chi_E(D_{\text{in}}). \quad (84)$$

If high-energy projection coherence suppresses two of the three paired directions, or gives them anomalous dispersion, then the effective exponent of  $D_{\text{in}}^{(E)}$  can decrease from three toward one even though the infrared reconstruction remains three-dimensional. Thus dimensional flow is attributed to scale-dependent inner-time coherence, not to a literal loss of the outer-time order.

*Remark 8.3* (Why outer time is not made fractal). The cone construction in [Section 7.3](#) and the stationary-action reconstruction in [Section 9](#) use a smooth coarse outer-time parameter. Making that parameter itself fractal would require replacing the local foliation, connection one-form, and short-tick expansion by a different nonlocal theory. The construction instead keeps outer time as the completed observational order and locates fractality in the inner spectral density.

## 9 Temporal kernels and stationary action

The spectral triple explains geometry through operator spectra. A physical theory also needs dynamics. The kernel layer supplies the missing bridge from inner-time re-creation to classical action.

### 9.1 From tick amplitudes to action

Let  $Q$  be a configuration manifold. A discrete DTT history is a sequence

$$q_\bullet = (q_0, q_1, \dots, q_N), \quad t_n = t_0 + n\varepsilon, \quad (85)$$

with fixed endpoints. A one-tick transition is represented by a  $J$ -phase kernel

$$K_\varepsilon(q_{n+1}, q_n) = a_\varepsilon(q_{n+1}, q_n) \exp\left(\frac{J}{\hbar} G_\varepsilon(q_n, q_{n+1})\right). \quad (86)$$

The  $N$ -tick amplitude is the composition integral

$$\mathcal{K}(q_N, q_0) = \int \prod_{n=0}^{N-1} K_\varepsilon(q_{n+1}, q_n) \prod_{n=1}^{N-1} dq_n. \quad (87)$$

The phase is additive:

$$G[q_\bullet] = \sum_{n=0}^{N-1} G_\varepsilon(q_n, q_{n+1}). \quad (88)$$

In the semiclassical limit, rapidly oscillating nonstationary histories cancel and leading contributions come from

$$\frac{\partial}{\partial q_n} (G_\varepsilon(q_{n-1}, q_n) + G_\varepsilon(q_n, q_{n+1})) = 0, \quad n = 1, \dots, N-1. \quad (89)$$

These are the discrete Euler-Lagrange equations.

If

$$G_\varepsilon(q, q + \varepsilon v) = \varepsilon L(q, v) + O(\varepsilon^2), \quad (90)$$

then the continuum limit gives the usual Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} = 0. \quad (91)$$

Thus Hamilton's principle is not imposed as a positive-cost minimization rule. It arises as stationary phase of composed inner-time amplitudes.

*Remark 9.1* (Fractal inner-time corrections). The derivation above assumes a local short-tick expansion and Markovian composition at the scale where classical mechanics is reconstructed. A fractal or recursive inner-time ultraviolet regime should therefore enter as a correction to this leading local theory, for example through memory terms

$$G[q_\bullet] = \sum_n G_\varepsilon(q_n, q_{n+1}) + \sum_{m>n} M_{m-n}(q_m, q_n) + \dots, \quad (92)$$

or through fractional/anomalous powers of the inner Dirac spectrum. Such corrections may be useful for ultraviolet phenomenology, but they are not part of the theorem-level reconstruction of ordinary stationary action.

## 9.2 Local normal form for mechanical kernels

The following theorem explains why ordinary finite-dimensional mechanics has the familiar metric-plus-one-form-minus-potential form.

**Theorem 9.2** (Local mechanical kernel normal form). *Assume that  $Q$  is finite-dimensional and that a one-tick phase  $G_\varepsilon(q, q + \varepsilon v)$  satisfies: locality, smoothness near the diagonal, covariance under coordinate changes, mechanical short-tick scaling, and non-degenerate quadratic stability in  $v$ . Then, up to endpoint gauge terms and higher-order corrections,*

$$G_\varepsilon(q, q + \varepsilon v) = \varepsilon \left[ \frac{1}{2} g_q(v, v) + A_q(v) - V(q) \right] + O(\varepsilon^2), \quad (93)$$

where  $g$  is a non-degenerate symmetric tensor,  $A$  is a one-form, and  $V$  is a scalar potential. The endpoint gauge transformation  $G_\varepsilon \mapsto G_\varepsilon + F(q_{n+1}) - F(q_n)$  acts as  $A \mapsto A + dF$  in the continuum.

*Proof sketch.* Locality and smoothness give a Taylor expansion near  $q' = q$ . Mechanical scaling separates terms by powers of  $r = q' - q = \varepsilon v$ . Non-degenerate quadratic stability makes the leading even quadratic term a metric-like tensor. Covariance forces the linear term in  $v$  to transform as a one-form and the zeroth-order term as a scalar. Endpoint differences shift only the one-form by an exact form and do not change fixed-endpoint equations.  $\square$

This result is important for the spectral-triple construction because it shows that DTT need not merely assume a Lagrangian. Under regularity assumptions, a natural Lagrangian is the generic leading continuum form of a local temporal phase kernel.

## 9.3 Hidden inner increments and Schur-complement inertia

Let  $r \in \mathbb{R}^n$  be an observed coordinate increment and  $u \in \mathbb{R}^k$  hidden inner-time increments. Consider a leading quadratic extended tick phase

$$G_\varepsilon(q; r, u) = \frac{1}{2\varepsilon} \left( r^T M r - 2r^T C u + u^T N u \right) + \Theta_a(q) r^a - \varepsilon V(q) + R_\varepsilon(q; r, u), \quad (94)$$

with  $N$  non-degenerate and  $R_\varepsilon$  higher order. The projected visible kernel integrates over  $u$ :

$$K_\varepsilon(q + r, q) = \int a_\varepsilon(q; r, u) \exp \left( \frac{J}{\hbar} G_\varepsilon(q; r, u) \right) d^k u. \quad (95)$$

Stationary elimination gives

$$u_* = N^{-1} C^T r \quad (96)$$

and the visible phase

$$G_\varepsilon^{\text{eff}}(q, q + r) = \frac{1}{2\varepsilon} r^T g_{\text{eff}} r + \Theta_a(q) r^a - \varepsilon V(q) + \dots, \quad (97)$$

where

$$g_{\text{eff}} = M - C N^{-1} C^T. \quad (98)$$

This is the Schur-complement mechanism for effective inertia. In a scalar Galilean inner-clock model, one obtains a mass parameter of the schematic form

$$m_{\text{eff}} = \frac{\kappa I_i}{I_i + \kappa \ell^2}, \quad (99)$$

where  $I_i$  is inner-clock stiffness,  $\kappa$  an outer channel stiffness, and  $\ell$  an outer-inner coupling scale. The same coefficient is both the curvature of the projected short-time phase and the central charge in the corresponding projective Galilei representation.

## 10 Mass, energy, and gauge structure

### 10.1 Mass as temporal frequency

If the Dirac spectrum contains frequencies  $\lambda_n$ , then the energy scale is

$$E_n = \hbar \lambda_n, \quad (100)$$

and a rest-mass scale can be assigned by

$$m_n = \frac{\hbar \lambda_n}{c^2}. \quad (101)$$

This is not yet a derivation of the observed particle spectrum. It is a spectral interpretation of mass: rest mass is the energy of persistent temporal recurrence. A nonzero spectral gap

$$\Delta = \inf (\text{Spec}(|D|) \setminus \{0\}) \quad (102)$$

implies a minimum mass scale

$$m_{\min} = \frac{\hbar \Delta}{c^2}. \quad (103)$$

### 10.2 Lacunary temporal spectra and the mass-hierarchy target

A stronger mass-hierarchy target is that the inner temporal spectrum may be lacunary or discretely self-similar, producing log-periodic clusters of Dirac eigenvalues. In a toy model one has gaps obeying a discrete scaling relation

$$\lambda_{n+k} \simeq r^{-1} \lambda_n, \quad (104)$$

so that spectral bands appear in geometric families. If a physical finite Dirac sector identifies selected eigenvalue gaps with rest energies, such clustering could imitate flavor hierarchies or generation structure.

This is a research target. The identification  $m \propto \hbar \lambda / c^2$  is meaningful for persistent recurrence modes, but observed fermion masses require a finite internal Dirac operator, mixing data, electroweak symmetry breaking, and renormalization. Koide-like relations, three-generation counting, and dark-sector band claims are therefore not part of the theorem-level results. The narrower claim is that a fractal or lacunary temporal spectrum supplies a natural source of hierarchical scales that can later be coupled to a full particle model.

### 10.3 Charge and internal phase

Temporal phase operators  $V_\theta$  implement  $U(1)$  rotations. Non-Abelian charges require finite multiplicity spaces and internal matrix algebras. In the NCG framework, the unitary group of the algebra, modulo central redundancies, determines gauge transformations. Thus a temporal derivation of the Standard Model would require a principled derivation of the finite algebra and representation data. This article only identifies where such a derivation would enter.

## 10.4 Tick-layer internal symmetry model

A useful model-level extension is to interpret the first three stable directions as spatial and the higher inner-time layers as internal symmetry fibers. The table below is a mnemonic for protected cluster multiplicities rather than a literal derivation from numbered ticks. The theorem-level target is spectral: rank 1, 2, and 3 inner-time eigenspace bundles with suitable reductions. This is not yet a derivation of the Standard Model representation content, but it gives a concrete target for the finite algebra  $A_F$  in [Sections 5](#) and [6](#).

The proposed hierarchy is:

| Tick layers                       | Role                    | Model symmetry                      |
|-----------------------------------|-------------------------|-------------------------------------|
| $t_r^{(0)}, t_r^{(1)}, t_r^{(2)}$ | spatial projection axes | quaternionic $\mathfrak{H} / SO(3)$ |
| $t_r^{(3)}$                       | phase layer             | $U(1)_Y$ -type phase                |
| $t_r^{(4)}, t_r^{(5)}$            | temporal doublet        | $SU(2)_L$ -type action              |
| $t_r^{(6)}, t_r^{(7)}, t_r^{(8)}$ | temporal triplet        | $SU(3)_C$ -type action              |

In this reading, a doublet

$$\psi_L = \begin{pmatrix} t_r^{(4)} \\ t_r^{(5)} \end{pmatrix} \quad (105)$$

transforms by  $\psi_L \mapsto U\psi_L$  with  $U \in SU(2)$ , while a triplet

$$\chi = \begin{pmatrix} t_r^{(6)} \\ t_r^{(7)} \\ t_r^{(8)} \end{pmatrix} \quad (106)$$

transforms by  $\chi \mapsto \exp(iT^a\theta_a)\chi$  with  $T^a \in \mathfrak{su}(3)$ . The combined model symmetry is then

$$G_{\text{tick}} = U(1)_Y \times SU(2)_L \times SU(3)_C, \quad (107)$$

or, after quotienting common centers in the usual Standard-Model manner,

$$G_{\text{phys}} \simeq \frac{SU(3)_C \times SU(2)_L \times U(1)_Y}{\mathbb{Z}_6}. \quad (108)$$

The NCG interpretation is that the finite internal algebra  $A_F$  should be reconstructed as the algebra of stable automorphisms of higher tick-layer fibers. The conservative statement is programmatic:

$$\text{higher temporal fibers} \implies A_F, \mathcal{H}_F, D_F, \mathcal{J}_{\text{NCG}}, \gamma_F. \quad (109)$$

Only after these finite data are derived can one claim an actual reconstruction of the Standard Model. Until then, the tick-layer table is a representation-theoretic ansatz and a guide for future work.

*Remark 10.1 (Symmetry breaking).* A possible temporal analogue of Higgs condensation is a coherent bilinear between the doublet layers, for example  $\Phi_H \sim \langle t_r^{(4)} t_r^{(5)} \rangle$ . A nonzero stable value could break  $SU(2)_L \times U(1)_Y$  to  $U(1)_{\text{EM}}$ . This is an interesting model target, but it must eventually reproduce the Higgs potential, Yukawa couplings, chiral representations, and anomaly cancellation conditions before it can be treated as physics rather than structural analogy.

## 10.5 Gauge curvature from inner fluctuations

For a temporal spectral triple, one-forms

$$A = \sum_i a_i [D, b_i] \quad (110)$$

measure changes in the Dirac spectrum under algebraic fluctuations. In DTT language, a gauge field is a coherent deformation of the relation between inner-time phases and outer-time propagation. The curvature  $F_A = dA + A^2$  measures the obstruction to flattening these deformations globally.

This provides a clean conceptual link between gauge theory and temporal ontology. However, it should be stated cautiously: reproducing the exact Standard Model gauge group, hypercharges, chirality, Higgs sector, and fermion generations is a separate and nontrivial task.

## 10.6 Protected inner-time bundles and adiabatic holonomy

There is a second, more spectral route to gauge structure that fits especially well with the scale-windowed operators of [Section 8.4](#). Let  $X_{\text{slow}}$  denote the emergent slow configuration manifold or observable chart, and let  $H_{\text{fast}}(x)$  be the inner-time operator seen at slow point  $x$ . Suppose an isolated spectral cluster has a smooth projector  $P_r(x)$  of constant rank  $r$ . Then

$$\mathcal{E}_r = \text{Ran } P_r \longrightarrow X_{\text{slow}} \quad (111)$$

is a Hermitian vector bundle. In a local orthonormal frame  $u_a(x)$  its adiabatic connection is

$$(\mathcal{A}_r)_{ab} = \langle u_a, du_b \rangle, \quad \mathcal{F}_r = P_r (dP_r)^2 P_r, \quad (112)$$

and the holonomy lies in  $U(r)$  [[27](#), [28](#)]. This is the Berry-Simon-Wilczek-Zee mechanism expressed inside the DTT inner-time spectral sector.

The minimal Standard-Model-like target can then be stated conditionally. A rank-one protected bundle gives a  $U(1)$  connection. A rank-two bundle with a preserved antiunitary quaternionic structure has structure group  $Sp(1) \simeq SU(2)$ . A rank-three bundle with a preserved complex volume form has structure group  $SU(3)$ . Hence the pattern

$$(r_1, r_2, r_3) = (1, 2, 3) \quad (113)$$

with these structure reductions yields

$$U(1) \times SU(2) \times SU(3) \quad (114)$$

as a holonomy group target. This statement is stronger than a visual tick-counting analogy but still weaker than a completed Standard Model derivation. It does not determine chirality, hypercharge assignments, anomaly cancellation, the Higgs sector, or Yukawa matrices.

Fractal inner-time spectra can enter this holonomy picture through the same windows used in [Section 8.4](#): the projectors  $P_r^{(E)} = \chi_E(D_{\text{in}}) P_r \chi_E(D_{\text{in}})$  may change rank, gap size, curvature norm, or leakage under scale variation. Thus dimensional flow, gauge emergence, and coupling corrections can be organized by one spectral object without claiming that all Standard-Model data have already been derived.

## 11 Algebraic stability of three-dimensional space

The stability criterion below is conditional: stable space is modeled by an algebraic interface that is both associative and non-commutative, and under the stated assumptions this interface selects the quaternionic vector sector.

### 11.1 Two requirements

Let  $\mathcal{A}_{\text{temp}}$  be the temporal operator algebra generated by inner shifts, inner generators, phase operators, and outer-time evolution.

1. **Associativity.** For all  $a, b, c \in \mathcal{A}_{\text{temp}}$ ,

$$(ab)c = a(bc). \quad (115)$$

This ensures unambiguous nesting of temporal re-creation operations.

2. **Non-commutativity.** There exist  $a, b \in \mathcal{A}_{\text{temp}}$  with  $[a, b] \neq 0$ . This supplies spectral discreteness and order sensitivity.

Associativity without non-commutativity tends toward classical commutative geometry. Non-commutativity without associativity threatens coherent composition. Stable temporal geometry therefore requires both.

### 11.2 Conditional stability criterion

**Theorem 11.1** (Associative non-commutative stability criterion). *Consider a DTT model in which extended spatial axes arise from paired inner-time modes, temporal compositions are represented by an associative operator algebra, and physically stable excitations require nonzero commutator norms and a nonzero infrared spectral gap. Then the minimal nontrivial stable spatial sector with three independent paired modes has spectral dimension  $d_s = 3$  and supports a coherent associative non-commutative temporal algebra of the crossed-product type.*

*Proof.* For three paired modes, the algebra  $\mathcal{S}(\mathbb{Z}^3) \rtimes \mathbb{Z}^3$  is associative and non-commutative through  $[N_a, U_b] = \delta_{ab}U_b$ . The Dirac operator has cubic Weyl growth and, after finite internal and bounded perturbations, preserves the spectral dimension. The nonzero commutators provide quantized spectral transitions, while associativity guarantees unambiguous composition of shifts, phases, and projections. Hence the three-pair sector satisfies the stated stability requirements.  $\square$

**Conjecture 11.2** (Minimality of the three-pair sector). *Within SMM/DTT, three paired inner-time modes are the minimal sector that simultaneously supports: (i) nontrivial stable extension in all observed spatial directions; (ii) associative temporal composition; (iii) nonzero spectral gaps; and (iv) closed rotational/gauge representations compatible with macroscopic three-dimensionality. Lower sectors are projectionally incomplete for observed stable matter, while higher primitive sectors either reduce to internal/gauge multiplicities or require additional constraints to avoid instability, non-associative closure, or unobserved extended dimensions.*

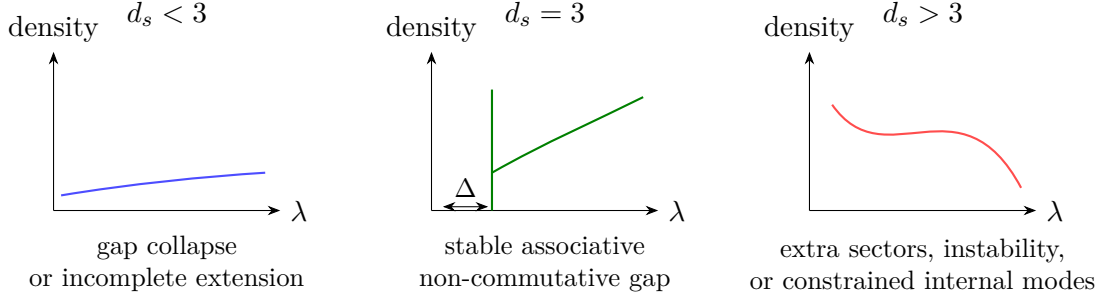


Figure 7: Stability picture. The robust sector is characterized by associative composition, nonzero commutator structure, and an infrared spectral gap. The claim that this sector is uniquely three-dimensional is presented as a conditional conjecture requiring further classification.

The conjecture is stronger than the theorem. It is the part that should be tested by further algebraic classification, numerical spectral models, and comparison with known constraints on rotations, division algebras, and Clifford modules.

### 11.3 Quaternionic cue and octonionic caution

The privileged role of three spatial dimensions resonates with the relation between quaternions,  $SU(2)$ , and rotations in three dimensions. Quaternions are associative and non-commutative. Octonions are non-associative and naturally appear in higher-dimensional exceptional structures. This does not prove that three dimensions are forced, but it supports the idea that associative non-commutative closure is especially natural in the three-dimensional sector.

## 12 Stable records and Born-form probabilities

The spectral triple and kernel layers describe phase and dynamics. Measurement and probability should not be used to generate those structures. They enter later through stable records.

### 12.1 Selected $J$ -linear event algebra

Once  $J$  is selected, the bounded  $J$ -linear algebra is

$$\mathcal{B}_J(H_R) = \{T \in \mathcal{B}(H_R) : TJ = JT\}. \quad (116)$$

Effects are positive contractions

$$\text{Eff}(H) = \{E \in \mathcal{B}_J(H_R) : 0 \leq E \leq I\}. \quad (117)$$

This event structure is non-Boolean. Thus the latent phase sector is not a hidden classical store of predetermined outcomes.

## 12.2 Phase-balanced composition

If two selected systems carry  $J_A$  and  $J_B$ , their physical composite identifies local phase rotations:

$$(J_A x) \otimes y \sim x \otimes (J_B y). \quad (118)$$

The balanced real tensor product is canonically the real form of the ordinary complex tensor product. This explains why independently selected systems share one quantum phase standard when they interact.

## 12.3 Stable record principle

A finite stable record experiment consists of an apparatus Hilbert space  $K$ , a ready state  $\eta_0$ , a unitary coupling  $W$ , and a finite pointer algebra generated by orthogonal idempotents  $R_i$  with  $\sum_i R_i = I_K$ . The preparation isometry is

$$V\psi = W(\psi \otimes \eta_0), \quad (119)$$

and the induced effects are

$$E_i = V^*(I \otimes R_i)V, \quad E_i \geq 0, \quad \sum_i E_i = I. \quad (120)$$

**Principle 12.1** (Finite stable-record Born map). *In a finite-dimensional selected complex sector, assume: every finite POVM is realized by a stable record dilation; probabilities depend only on completed record-effect classes; probabilities are normalized and additive on disjoint pointer idempotents; and classical randomization gives affine mixing of preparations. Then each preparation is represented by a density operator  $\rho$  and*

$$\mu(E) = \text{Tr}(\rho E) \quad (121)$$

for all finite-dimensional effects  $E$ . In particular,

$$\text{Pr}(i|\rho) = \text{Tr}(\rho E_i). \quad (122)$$

The stable-record layer is included because it prevents a common misunderstanding: DTT does not require consciousness to collapse the wavefunction. Phase is selected before measurement, and collapse is an observer-accessible diagonal approximation after stable record formation.

## 13 Comparison with other emergent-spacetime programs

Table 2: Conceptual comparison with other emergent-spacetime programs. The table is schematic and indicates the role emphasized by each approach.

| Framework            | Time? | Spectral? | Geometry?                    | Dimension?  |
|----------------------|-------|-----------|------------------------------|-------------|
| Loop quantum gravity | No    | Partial   | Emergent spectra of geometry | Not primary |
| Causal set theory    | Yes   | Limited   | Emergent from order          | Not unique  |

|                                |     |          |                                         |                                            |
|--------------------------------|-----|----------|-----------------------------------------|--------------------------------------------|
| Tensor networks and holography | No  | Indirect | Emergent from entanglement              | Model-dependent                            |
| Group field theory             | No  | Yes      | Condensate geometry                     | Condensate-dependent                       |
| Quantum graphity               | No  | Limited  | Emergent locality                       | Phase-dependent                            |
| Connes spectral action         | No  | Yes      | Spacetime plus finite internal geometry | No                                         |
| SMM/DTT spectral triples       | Yes | Yes      | Emergent from temporal spectrum         | Cycle-pair growth plus stability criterion |

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Loop quantum gravity quantizes geometric operators through holonomy-flux algebras and spin networks [14, 5]. Causal set theory makes causal order primary but does not usually formulate geometry through spectral triples [15]. Tensor-network and holographic approaches reconstruct geometry from entanglement patterns [16]. Group field theory and tensor models describe spacetime as a condensate of pre-geometric quanta [17]. Quantum graphity models locality as a phase of an underlying graph [18].

The SMM/DTT approach is closest in formal language to NCG, but its philosophical direction differs from the Connes-Chamseddine spectral action. In the spectral action program, spacetime geometry and finite internal geometry are input data whose spectral action yields gravity and gauge theory. In DTT, the temporal algebra is the source from which spatial geometry is intended to emerge. The price of this ambition is that more derivations are required; the advantage is that spacetime, quantum phase, action, and records can be placed in one temporal reconstruction chain.

## 14 Phenomenological and mathematical predictions

The framework suggests several concrete directions for analysis and possible testing.

### 14.1 Spectral gaps and minimum mass scales

If the temporal Dirac operator has a nonzero infrared gap  $\Delta$ , then no persistent excitation exists below

$$m_{\min} = \frac{\hbar\Delta}{c^2}. \quad (123)$$

This could be compared with neutrino scales, dark-sector thresholds, or analogue-system gap measurements. The theory does not yet predict the numerical value of  $\Delta$ ; it identifies the type of spectral datum that would be needed.

### 14.2 Fractal inner-time dimensional flow

The scale-dependent inner spectral dimension

$$d_{\text{in}}(\sigma) = -2 \frac{\text{d} \log \text{Tr} e^{-\sigma D_{\text{in}}^2}}{\text{d} \log \sigma} \quad (124)$$

should flow to 3 in the stable infrared sector. The effective spacetime dimension is then

$$d_{\text{eff}} = 1 + d_{\text{in}}, \quad (125)$$

so a high-energy flow  $d_{\text{in}} : 3 \rightarrow 1$  appears observationally as  $d_{\text{eff}} : 4 \rightarrow 2$ . Possible signatures include modified dispersion relations, anomalous density of states, nonstandard entropy scaling, vacuum-fluctuation suppression, or weak energy-dependent completion delays through  $\Theta(E)$ ,  $\rho(E)$ , and  $\kappa(E)$ .

### 14.3 Correlated gauge-running and log-periodic corrections

If gauge sectors arise from protected inner-time bundles over the same temporal spectrum, then high-scale corrections to their inverse couplings should not be arbitrary independent functions. A conservative parametrization is

$$g_a^{-2}(\mu) = g_{a,\text{SM}}^{-2}(\mu) + \eta_a \Phi(\mu/E_*) + \text{extthresholdterms}, \quad a = 1, 2, 3, \quad (126)$$

where the profile  $\Phi$  is common because it is controlled by the same inner heat trace or spectral-decimation law, while the amplitudes  $\eta_a$  depend on the sector projector. If the inner temporal spectrum is exactly or approximately self-similar, a possible correction has the schematic form

$$\Phi(x) = x^{-q} (1 + \epsilon \cos(\omega \log x + \phi) + \dots). \quad (127)$$

The logarithmic oscillation would be a distinctive discrete-scale-invariance signature; ordinary heavy-threshold corrections would not usually produce a common log-periodic phase across all three sectors. This is a phenomenological target, not yet a prediction with fixed numerical coefficients.

### 14.4 Tick aliasing

If the inner tick time  $\varepsilon$  is physical, then the logarithm of a one-tick operator is branch-dependent above

$$E_{\text{alias}} \sim \frac{\pi \hbar}{\varepsilon}. \quad (128)$$

This predicts either a consistency bound on physical spectra or a possible high-energy periodicity/aliasing effect. If  $\varepsilon$  is a regulator, aliasing disappears in the continuum limit.

### 14.5 Analogue simulations

A lattice system implementing

$$[N_a, U_b] = \delta_{ab} U_b \quad (129)$$

can simulate the temporal algebra. Deformations that reduce commutator norms or spoil associative composition should lead to spectral-gap collapse, excess dephasing, or anomalous density-of-states scaling. Photonic lattices, superconducting circuits, and synthetic gauge systems are natural platforms.

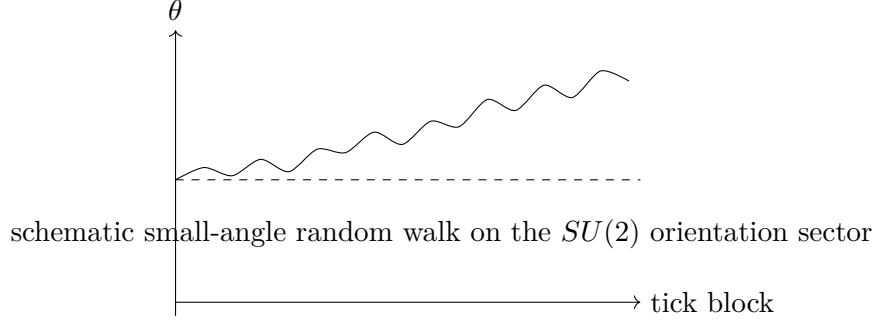


Figure 8: Toy small-angle rotational diffusion trace generated by unbiased inner-time orientation updates. The proposed experimental signature is not this particular realization but the  $SU(2)$  Brownian generator and the random-walk spectrum  $S_\theta(f) \simeq 2D/(2\pi f)^2$ .

## 14.6 Rotational diffusion as a falsifiable quaternionic signature

The quaternionic route predicts that the stable orientation sector is governed by  $SU(2)$ , the unit-quaternion group. If microscopic inner-time updates contain small unbiased orientation errors, the large-block limit should be Brownian motion on  $SU(2)$  [24] rather than on an arbitrary rotation group. In a local small-angle chart, write

$$U_{n+1} = \exp\left(\sqrt{2D\delta t} \xi_n^a \frac{\sigma_a}{2}\right) U_n, \quad (130)$$

where  $\xi_n^a$  are independent unit-variance random variables,  $\sigma_a$  are Pauli generators, and  $D$  is an effective rotational diffusion constant. The continuum generator is

$$\mathcal{L}_{\text{rot}} = D\Delta_{SU(2)}. \quad (131)$$

For a small angular coordinate  $\theta(t)$  this gives the random-walk spectrum

$$S_\theta(f) \simeq \frac{2D}{(2\pi f)^2}. \quad (132)$$

Thus precision gyroscopes, atom interferometers, resonant optical cavities, spin-precession experiments, and superconducting phase devices can bound  $D$  even when they cannot directly observe the inner-time substrate. The key discriminator is not generic noise, but an  $SU(2)$ -structured, orientation-sector random walk with the predicted low-frequency scaling.

## 14.7 Record-defect bounds

The stable-record layer suggests quantitative record-defect measures. If  $\mathcal{R}$  is a pointer algebra and  $\rho$  a state, the off-diagonal record defect may be modeled by

$$\epsilon_{\text{off}}(\rho; \mathcal{R}) = \|\rho - \Delta_{\mathcal{R}}(\rho)\|_1, \quad (133)$$

where  $\Delta_{\mathcal{R}}$  is dephasing in the pointer algebra. Interference is suppressed for an observer when this defect is below the apparatus resolution. This gives a route to connecting DTT language with standard decoherence estimates.

## 15 Connections to foundational open problems

This section is deliberately cautious. The framework suggests new formulations of famous problems, but it does not solve them here.

### 15.1 Yang-Mills mass gap

A temporal gauge sector with nonzero commutator norms and compact inner-time modes may produce a non-perturbative spectral gap. A research-level conjecture is

$$m_{\text{gap}} \sim \alpha \omega_0 \quad (134)$$

for an appropriate compactification frequency  $\omega_0$  and dimensionless coupling  $\alpha$ . To become a solution of the Yang-Mills mass-gap problem, one would need a rigorous construction of a four-dimensional quantum Yang-Mills measure, proof of Osterwalder-Schrader/Wightman axioms, and a positive mass gap in gauge-invariant correlation functions. The framework only identifies a candidate spectral mechanism.

### 15.2 Riemann zeta spectra

The inner-time generator defines zeta-type spectral functions

$$\zeta_D(s) = \text{Tr} |D|^{-s} \quad (135)$$

where defined. A Hilbert-Polya-type program would seek a self-adjoint operator whose spectrum corresponds to the imaginary parts of nontrivial zeta zeros. DTT suggests interpreting arithmetic oscillations as phase recurrences in a temporal spectral system. This is a speculative bridge, not a proof of the Riemann hypothesis.

### 15.3 Black-hole information

If black-hole entropy is interpreted as degeneracy of temporal spectral modes below a critical scale, one may write schematically

$$S \sim \log \#\{n : \lambda_n \leq \lambda_*\}. \quad (136)$$

A unitary temporal spectrum could in principle store information in modes not visible as classical spatial data. Making this precise would require coupling the temporal triple to semiclassical gravity and reproducing Hawking radiation, Page curves, and horizon thermodynamics.

### 15.4 Cosmological constant

Temporal phase variance suggests a schematic residual

$$\Lambda_{\text{eff}} \propto \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \theta_n^2 \quad (137)$$

for suitable renormalized phases. This may offer a route to explaining a small positive vacuum curvature as a global phase imbalance rather than a sum of independent field zero-point energies. A genuine derivation would require a covariant spectral action calculation and a numerical prediction.

## 16 Limitations and future work

Several essential problems remain open.

1. **Lorentzian spectral geometry.** Standard spectral triples are Euclidean. DTT uses a hyperbolic inner/outer temporal distinction. A fully Lorentzian or Krein-spectral model is needed.
2. **Exact Standard Model data.** The protected-bundle mechanism gives a conditional gauge-holonomy target, but the finite algebra, fermion representation, chirality, Higgs sector, hypercharges, anomaly cancellation, and Yukawa matrices have not been derived from inner-time principles.
3. **Numerical constants and masses.** The framework does not derive the observed values of  $\hbar$ ,  $c$ ,  $G$ , particle masses, gauge couplings, or the cosmological constant. Lacunary spectra provide a hierarchy mechanism only after a physical finite Dirac sector and matching prescription are supplied.
4. **Interacting quantum field theory.** Renormalization, gauge fixing, anomalies, and continuum limits remain to be developed.
5. **Universal stable record dilation.** The Born-form record theorem assumes that all finite effects can be generated by stable records; deriving this from inner-time dynamics remains open.
6. **Dimensional minimality.** The three-pair model gives  $d_s = 3$ , but the claim that three is uniquely stable needs a deeper classification theorem.
7. **Microscopic fractal recursion.** The bounded interpolation for  $d_{\text{in}}(E)$ , possible log-periodic corrections, and protected-cluster leakage should be computed from a temporal transfer operator or recursive spectral-decimation model rather than inserted phenomenologically.

Promising next steps include finite-matrix simulations of temporal triples, heat-trace calculations for fractal inner-time dimensional flow, derivation of the interpolation  $d_{\text{eff}} = 1 + d_{\text{in}}$  from microscopic recursive spectra, construction of protected rank-1,2,3 inner-time bundles and their adiabatic curvatures, analogue implementations of the shift-generator algebra, numerical tests of rotational diffusion, development of lacunary spectral models coupled to finite Dirac operators, rigorous development of the non-compact/semifinite outer-time model, and regulator-level matching theorems for gauge couplings and mass parameters.

## 17 Conclusion

The SMM/DTT spectral-triple framework presents a layered route from temporal ontology to physical structure. It begins with a real Hilbert substrate, identifies admissible quantum phase operators in the Real commutant, selects the physical phase by positive-time polar decomposition, uses quaternionic stable-composition criteria to motivate three spatial directions, constructs temporal operator algebras and spectral triples, extracts geometry from spectral growth, adds a fractal inner-time refinement in which  $d_{\text{eff}} = 1 + d_{\text{in}}$  can flow from four toward two, keeps  $d_H^{\text{in}}$ ,  $d_{\text{in}}$ , and  $d_g$

conceptually distinct, reconstructs stationary action from local  $J$ -phase kernels, interprets inertia through hidden inner-time Schur complements, formulates a cone-first route to Lorentzian geometry and gravitational infrared universality, adds a protected-bundle holonomy route to gauge structure, and locates measurement probabilities in downstream stable records.

The main technical features are the corrected treatment of bounded algebras versus unbounded generators, the compact/non-compact distinction for spectral triples, the conditional quaternionic dimensional-selection theorem, the spectral explanation of dimension through three paired inner modes, the finite-regulator eigenvalue checks, the disciplined spectral meaning of fractal inner time, the integration of kernel dynamics, the protected-cluster gauge interface, and the explicit separation between Euclidean spectral triples and Lorentzian realization models. The framework remains ambitious and conditional, but its claims are separated by status: some are theorem-level consequences of stated assumptions, some are model-level derivations, and some are conjectural research directions.

The article should therefore be read not as a final theory of everything but as a disciplined reconstruction program: physical space, classical action, gauge structure, and observer records are not independent primitives, but successive projections of a deeper temporal spectral order.

## A Projection-history normalization and the structural integer 137

This appendix records a structural route from the three-dimensional projection hierarchy to the integer 137 in a cautious form because it is attractive but potentially easy to overstate. The claim is not that the measured fine-structure constant has been derived. The claim is that a specific projection-history package, once its hypotheses are imposed, has a unique bare structural value 137.

### A.1 Projection-history package

The admissible package contains four ingredients.

1. A three-dimensional projection stage selected by the quaternionic stability filter of [Theorem 3.2](#).
2. A height-three finite closure hierarchy  $P = (P_0, P_1, P_2, P_3)$  with maximal closure multiplicity

$$M(P) = 10. \tag{138}$$

A canonical realization is the quadratic closure flag

$$P_d = \dim \operatorname{Sym}^2(E_d) = \binom{d+2}{2}, \quad d = 0, 1, 2, 3, \tag{139}$$

giving

$$1, 3, 6, 10. \tag{140}$$

3. A branch-count reconstruction functor whose height- $n$  polynomial is

$$N_n(\lambda) = \sum_{d=0}^n (2^d - 1) \lambda^{n-d}. \tag{141}$$

At height three this gives

$$N_3(\lambda) = \lambda^2 + 3\lambda + 7. \tag{142}$$

4. A positive-energy real-Hilbert realization containing an exact  $J$ -linear rank-one tower

$$0 \longrightarrow K_d \longrightarrow K_{d+1} \longrightarrow \mathbb{C} \longrightarrow 0, \quad d = 0, 1, 2, \quad (143)$$

with  $\dim_J K_0 = 1$ .

The rank-one tower gives by additivity

$$\dim_J K_d = d + 1, \quad d = 0, 1, 2, 3, \quad (144)$$

and therefore the complex trace scale is

$$\lambda_{\text{op}} = \text{Tr}_J(I_K) = 1 + 2 + 3 + 4 = 10. \quad (145)$$

The operator-admissible valuation is thus  $W_{3,10}$  and

$$N_3(\lambda_{\text{op}}) = N_3(10) = 10^2 + 3 \cdot 10 + 7 = 137. \quad (146)$$

Moreover,  $N_3(\lambda) = 137$  has the unique positive solution  $\lambda = 10$  because

$$\lambda^2 + 3\lambda - 130 = (\lambda - 10)(\lambda + 13). \quad (147)$$

## A.2 Separation of the three appearances of ten

The framework contains several appearances of 10, and they should not be conflated.

| Source of ten             | Formula                        | Status                                 |
|---------------------------|--------------------------------|----------------------------------------|
| Closure multiplicity      | $M(P) = 10$                    | readout hypothesis/theorem target      |
| Quadratic flag            | $1 + 3 + 6 = 10$ or $P_3 = 10$ | canonical realization                  |
| Operator tower            | $1 + 2 + 3 + 4 = 10$           | scale-selection theorem within package |
| Sexagesimal compatibility | $10 \cdot 6 \cdot 6 = 360$     | interpretive compatibility             |

The branch coefficients 0, 1, 3, 7 also have a distinct role: they count nonempty subsets at each level and determine the polynomial  $N_3$ , not the operator scale.

## A.3 Relation to the fine-structure constant

A physical coupling would require additional analytic input:

1. a finite regulator or continuum limit;
2. positivity/reflection-positivity or transfer-matrix reconstruction;
3. a distinguished observable electromagnetic sector;
4. renormalization from the bare structural value to the measured low-energy coupling;
5. a computable correction term.

Therefore the only claim appropriate for the spectral-triple framework is:

Inside the stated projection-history package, the bare structural valuation gives 137 uniquely. Whether this structural integer is physically matched to  $\alpha^{-1}$  requires a separate coupling-matching theorem.

This appendix shows how the dimensional selection, real-Hilbert positive phase, and projection-history normalization programs can be made compatible without turning the main text into a numerological argument.

## B Notation and symbol glossary

| Symbol             | Meaning                                                                                 |
|--------------------|-----------------------------------------------------------------------------------------|
| $j$                | Split-complex temporal unit, $j^2 = +1$ , used for inner/outer time notation.           |
| $J$                | Orthogonal complex structure/quantum phase operator, $J^2 = -\text{Id}$ .               |
| $H_R$              | Real Hilbert substrate.                                                                 |
| $\mathcal{A}_R(U)$ | Real commutant of an orthogonal representation.                                         |
| $N_a$              | Inner-time number/frequency generator for paired mode $a$ .                             |
| $U_a$              | Shift operator for paired mode $a$ .                                                    |
| $D$                | Temporal Dirac operator.                                                                |
| $\mathcal{A}_0$    | Smooth dense temporal algebra of bounded operators.                                     |
| $d_s$              | Spectral dimension from eigenvalue or heat-kernel growth.                               |
| $d_H^{\text{in}}$  | Hausdorff/counting dimension of a possible self-similar inner-time support.             |
| $d_{\text{in}}$    | Running inner-time spectral dimension.                                                  |
| $d_g$              | Effective gravitational or Gauss-law dimension, not assumed equal to $d_{\text{in}}$ .  |
| $d_{\text{eff}}$   | Effective spacetime spectral dimension, organized here as $1 + d_{\text{in}}$ .         |
| $E_*$              | Crossover scale for possible inner-time dimensional flow.                               |
| $P_r^{(E)}$        | Scale-windowed projector onto a protected rank- $r$ inner-time spectral cluster.        |
| $\Theta(E)$        | Scale-dependent completion delay in the multiscale cone refinement.                     |
| $G_\varepsilon$    | One-tick phase function.                                                                |
| $K_\varepsilon$    | One-tick transition kernel.                                                             |
| $g_{\text{eff}}$   | Visible effective metric or inertia tensor after hidden inner variables are eliminated. |
| $\Delta$           | Infrared spectral gap of $ D $ .                                                        |
| $\mathcal{A}_r$    | Adiabatic connection on a protected inner-time eigenspace bundle.                       |

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