

Duality of Time and Emergent Gauge Structure from Inner-Time Spectral Geometry

Mohamed Haj Yousef

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Abstract

We present a mathematically explicit reformulation of the Duality of Time (DoT) program in which observable physics is obtained by projection from a dual-time arena rather than assumed on a primordial spacetime background. The framework starts from a product temporal space $T = \mathbb{R}_t \times F$, where t is the observable outer time and F is a compact inner-time space carrying a regular Dirichlet form and, when available, a compatible spectral triple. Physical states are defined on T and observables are obtained by spectral projection onto a low-resolution sector. Within this setting we prove a gauge-emergence theorem: adiabatic transport of protected eigenspace bundles of the inner-time operator produces a $U(m)$ connection on each rank- m cluster, and the minimal multiplicity pattern (1, 2, 3) together with natural structure-group reductions yields the effective gauge group $U(1) \times SU(2) \times SU(3)$. The Abelian sector is identified with hypercharge before symmetry breaking, while the non-Abelian sectors are identified with weak isospin and color. We derive a well-posed projected dynamics, formulate renormalized coupling functionals in terms of inner-time spectral data, and show how anomalous scaling or log-periodic corrections can arise when F has fractal spectral asymptotics. Gravity enters not as a fourth gauge factor but as the metric content of the projected spectral geometry. In this sense the framework yields a qualified geometric unification: couplings, gauge holonomy, gravity, and the qualitative organization of generations all arise from distinct structures of the same projected dual-time geometry. The same hierarchy suggests a matter-sector extension in which replicated spectral windows of fixed gauge-bundle type account for common charges across generations, while spectral level and overlap with the symmetry-breaking sector control masses and mixing. The paper deliberately separates theorem-level results from conjectural extensions: exact numerical values of the couplings, exact generation count, chirality, symmetry breaking, CKM/PMNS data, and fermion masses are not claimed here. The result is a submission-grade mathematical framework for DoT in which the central gauge-structure claim is stated and proved under explicit hypotheses. We also record a minimal explicit realization of the low-energy geometric sector: taking the inner-time space to be $SU(2) \cong S^3$ with its natural Dirichlet and Dirac-type spectral data yields a protected quaternionic three-direction block, so the three-generator observable chart is exhibited concretely rather than left as a bare hypothesis.

Keywords: duality of time; emergent gauge symmetry; spectral triples; fractal geometry; adiabatic holonomy; Dirichlet forms; mathematical physics; force unification; fermion generations; flavor hierarchy

1. Introduction

The gauge sector of the Standard Model is built on the symmetry group

$$G_{\text{SM}} = U(1)_Y \times SU(2)_L \times SU(3)_c,$$

with electromagnetism emerging only after electroweak symmetry breaking. This structure is extraordinarily successful phenomenologically, but its origin remains an open conceptual question. Why should nature select precisely these groups? Why do the observable couplings exhibit a hierarchy? How should gravity fit into the same picture?

The Duality of Time (DoT) program approaches these questions from a different starting point. Instead of treating space as fundamental and time as a single external parameter, it postulates that physical evolution is governed by two coupled temporal levels: an outer time that orders observable events and an inner temporal structure responsible for their generation. In recent work, a mathematically consistent dual-time manifold, independent temporal connections, curvature, and projection mechanism were developed for non-equilibrium systems, with a representation theorem showing how non-Markovian single-time equations can arise from local evolution on a bi-temporal domain [1]. Earlier foundational work described the broader physical intuition of complex time and the emergence of space from inner temporal dynamics [2].

The earlier force-unification draft that motivates the present paper made a bold claim: the gauge groups $U(1)$, $SU(2)$, and $SU(3)$ arise from the geometry of inner time. Its weakness was not the ambition but the missing mathematics. The state space was not fully defined, the projection operator was not specified as an operator-theoretic object, the claimed gauge emergence was suggestive rather than theorem-level, and exact numerical claims were presented without a renormalized framework. The purpose of the present paper is to replace that informal structure with a precise mathematical model.

The key move is to combine three established tools.

First, we use the dual-time manifold viewpoint developed in [1] as the kinematic basis for the theory. Second, we model inner time by an operator-algebraic or Dirichlet-form geometry, drawing on noncommutative geometry, spectral triples, and analysis on fractals [3-7]. Third, we derive the gauge sector from adiabatic holonomy: following Simon, Berry, and Wilczek-Zee, isolated or degenerate eigenspaces of a family of fast operators carry natural Abelian or non-Abelian connections under adiabatic transport [8,9].

The main result of this paper can be stated informally as follows. If the projected observable sector of DoT contains three protected inner-time eigenspace bundles of ranks 1, 2, and 3, where the rank-2 bundle carries a quaternionic structure and the rank-3 bundle carries a preserved complex volume form, then the effective gauge group of the projected theory is exactly

$$U(1) \times SU(2) \times SU(3).$$

This is a genuine theorem inside the model, not a metaphor. The theorem does not by itself derive chiral matter, the Higgs mechanism, or the observed numerical values of the couplings. Those remain open problems. But it does isolate a mathematically controlled mechanism by which the Standard Model gauge symmetry can emerge from dual-time structure. The broader unification claim of the present paper is correspondingly geometric rather than grand-unified: spectral data control couplings, adiabatic holonomy controls gauge structure, and projected metric data control gravity. The same architecture also suggests, at the level of a controlled hypothesis, how repeated fermion generations with common charges but different masses can arise from spectral replication.

The paper is organized as follows. Section 2 states the DoT postulate and defines the mathematical data. Section 3 introduces the dual-time state space, the projected observable algebra, and the projected

dynamics. Section 4 proves the gauge-emergence theorem and its $U(1)$, $SU(2)$, and $SU(3)$ corollaries. Section 5 derives the effective action and the coupling functionals. Section 6 explains how gravity enters as projected geometry rather than as an independent gauge factor, formulates the resulting qualified unification of forces, and relates the geometric hierarchy to generations, charges, and masses. Section 7 lists four falsifiable consequences and Section 8 clarifies the scope and limitations of the current framework. Section 2 now also records a minimal explicit $SU(2)$ -based realization of the low-energy three-direction sector, linking the present spectral-projection framework to the companion quaternionic route to 3D emergence.

2. The DoT postulate and the mathematical data

2.1. The DoT postulate

Postulate 2.1 (Duality of Time). There exists a fundamental dual-time arena

$$\mathcal{T} = \mathbb{R}_t \times F,$$

where $t \in \mathbb{R}$ is the outer time parameter that labels observable evolution, F is a compact inner-time space encoding generative temporal structure, physical states are defined on $\mathcal{T}$ rather than on an a priori spacetime manifold, and observable space, gauge structure, and effective dynamics arise only after projection from the full dual-time description to a coarse observable sector.

This postulate is the precise version of the earlier informal claim that space is a shadow of deeper temporal structure. The word *fractal* will mean, throughout this paper, that the inner-time space may possess anomalous spectral asymptotics or noninteger Hausdorff dimension. None of the core theorems below require noninteger dimension; fractality enters in the coupling asymptotics and high-energy corrections.

2.2. Inner-time geometry

We assume that F is a compact metric-measure space (F, d, μ) endowed with a regular conservative Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(F, \mu)$ and associated nonnegative self-adjoint generator L_F . This includes smooth compact manifolds as a special case and also a large class of fractal examples [5,6].

When available, we further assume that the inner-time geometry admits a spectral triple

$$(\mathcal{A}_F, \mathcal{H}_F, D_F),$$

where $\mathcal{A}_F$ is a dense involutive algebra represented on the Hilbert space $\mathcal{H}_F$, and D_F is a self-adjoint Dirac-type operator. In the smooth case L_F may be taken comparable to D_F^2 . In general, the Dirichlet form and the spectral triple are two compatible analytic descriptions of the same inner-time geometry.

The choice of a Dirichlet form is important for rigor. It provides a well-defined self-adjoint operator, a heat semigroup, and a notion of derivation even on singular spaces. By the theorem of Cipriani and Sauvageot, one can construct a Hilbert bimodule of one-forms and a closed derivation

$$\delta: \mathcal{A}_F \rightarrow \mathcal{H}_F^{(1)}$$

such that, on a suitable core,

$$L_F = \delta^* \delta.$$

This is the operator-theoretic replacement of the earlier informal language of “inner-time arrows”.

2.3. States and observables

Let $\mathcal{H}_{\text{ext}}$ be a separable Hilbert space carrying the external or slow degrees of freedom, and define the full state space

$$\mathcal{H} = L^2(\mathbb{R}_t, dt; \mathcal{H}_{\text{ext}} \otimes \mathcal{H}_F).$$

A physical state is a square-integrable map

$$\Psi: \mathbb{R}_t \rightarrow \mathcal{H}_{\text{ext}} \otimes \mathcal{H}_F.$$

The observable sector is defined by a spectral cutoff. For $\Lambda > 0$, let

$$P_\Lambda = \chi_{[0, \Lambda]}(L_F)$$

be the spectral projection onto the low-resolution inner-time sector. We then define the observable algebra at scale Λ by

$$\mathcal{A}_{\text{obs}, \Lambda} = (I \otimes P_\Lambda)(\mathcal{B}(\mathcal{H}_{\text{ext}}) \bar{\otimes} \mathcal{A}_F)(I \otimes P_\Lambda).$$

Expectation values of observable operators are computed in the projected states

$$\Psi_\Lambda = (I \otimes P_\Lambda)\Psi.$$

This gives a concrete mathematical meaning to the DoT claim that outer-time observables arise by projection from a deeper temporal structure.

2.4. Emergent spatial coordinates

The present paper does not claim to *derive* three dimensions from no input at all. Instead, it isolates the minimal low-energy hypothesis needed to recover them.

Assumption 2.2 (Three-generator observable chart). Below the cutoff Λ_s , the projected cotangent module generated by the derivation δ has rank 3 on the observable sector. Equivalently, there exist three self-adjoint elements

$$X^1, X^2, X^3 \in \mathcal{A}_{\text{obs}, \Lambda_s}$$

whose differentials span the low-energy one-form module, and no fourth independent generator appears below the same scale.

For any normal state ρ on $\mathcal{A}_{\text{obs}, \Lambda_s}$ we then define the effective spatial coordinates by

$$x^i(\rho) = \rho(X^i), \quad i = 1, 2, 3.$$

This assumption is the rigorous analogue of the earlier claim that three independent inner-time directions generate the observed spatial dimensions. The remainder of the paper works on the resulting local observable manifold M coordinatized by (x^1, x^2, x^3) and outer time t .

2.4.1. A minimal explicit quaternionic inner-time model

A concrete realization of Assumption 2.2 is obtained by choosing the inner-time space to be the compact Lie group $F_{\min} = \text{SU}(2) \cong S^3$ with normalized Haar measure dg . Let L_1, L_2, L_3 denote the left-invariant derivations corresponding to an orthonormal basis of $\mathfrak{su}(2)$, so that $[L_a, L_b] = 2\epsilon_{abc} L_c$. Define the inner-time Dirichlet form by $E(f) = \sum_{a=1}^3 \int_{F_{\min}} |L_a f|^2 dg$ and the associated generator by $\Delta_F = -\sum_{a=1}^3 L_a^2$. When one wants a spectral-triple realization, one may take $H_F = L^2(\text{SU}(2), \mathbb{C}^2)$ and a Dirac-type operator $D_F = -i \sum_{a=1}^3 \sigma_a L_a$, with σ_a the Pauli matrices. This choice is fully compatible with the dual-time kinematics of the dual-time manifold paper [1]: the full arena remains $T = R_t \times F$, while outer observables arise only after projection from the inner-time sector.

The advantage of this model is that the quaternionic structure is no longer heuristic. Unit quaternions act on H by conjugation, their Lie algebra is generated by the three imaginary directions, and the associated real adjoint module is $\text{Im}(H) \cong \mathbb{R}^3$. In the companion DTT derivation, associativity excludes the octonionic route and the unique associative noncommutative envelope is H , so the emergent mode space is precisely $W \cong \text{Im}(H) \cong \mathbb{R}^3$ [14]. The present paper uses the spectral counterpart of the same quaternionic structure.

2.4.2. Spectral block and realization of the three-direction sector

On $F_{\min}$ there is a canonical $\text{SU}(2)$ -covariant three-dimensional block in the one-form sector, namely the span of the left-invariant Maurer-Cartan one-forms $\sigma_1, \sigma_2, \sigma_3$ dual to L_1, L_2, L_3 . Equivalently, in the operator picture one may work with the three quaternionic generators J_1, J_2, J_3 acting on H . This block transforms irreducibly under the adjoint action and is the natural spectral seed of the emergent direction space. Choosing the observable cutoff Λ so that the projected derivation keeps exactly this three-dimensional block produces a projected cotangent module of rank 3. In that sense, the present model realizes the three-generator observable chart explicitly rather than merely assuming it.

If one prefers coordinate language, the spatial coordinates are then defined exactly as in Assumption 2.2: locally one chooses three self-adjoint projected observables X^1, X^2, X^3 whose differentials span the retained block, and for any normal state ω one sets $x^a(\omega) = \omega(P_\Lambda X^a P_\Lambda)$. The point is not that $\text{SU}(2)$ itself is already observable space; rather, its low-energy spectral data furnish the minimal internal triplet from which the observable three-direction sector is reconstructed after projection.

2.4.3. Relation to the gauge multiplicities (1, 2, 3)

This explicit spatial triplet should not be conflated with the rank-3 color bundle of Section 4. The spatial sector is a real three-dimensional direction module $\text{Im}(H) \cong \mathbb{R}^3$, whereas the color sector is a complex rank-3 protected eigenspace bundle of the fast operator. Likewise, the weak sector is the place where the quaternionic structure appears directly as a reduction $U(2) \rightarrow \text{SU}(2)$. The benefit of the minimal $\text{SU}(2)$ model is therefore conceptual clarity: one common inner-time spectral geometry can support a real spatial triplet, a quaternionic weak doublet, and further protected spectral clusters without identifying these sectors with one another.

Accordingly, the claim advanced here is modest and explicit. The $\text{SU}(2)$ model does not by itself derive the full multiplicity pattern (1, 2, 3) from first principles. What it does show is that the most delicate part of the geometric picture - the appearance of exactly three observable directions from inner-

time structure - admits a minimal operator-theoretic realization that matches the companion quaternionic 3D-emergence route [14].

2.5. Relation to extra-time constructions

The DoT framework is conceptually related to other multi-time ideas, including two-time physics [10], but it differs in a crucial respect. In DoT only the outer parameter t generates observable time evolution. The inner-time variable is not a second macroscopic time along which observers propagate. Instead, it enters through a fast operator on an internal Hilbert space and influences observables only after projection. This distinction is what allows the model to avoid introducing an additional directly observable timelike direction.

3. Dual-time dynamics and the projected observable theory

3.1. The total Hamiltonian

Let H_{ext} be a self-adjoint operator on $\mathcal{H}_{\text{ext}}$ and let $V(x)$ be a family of bounded or relatively bounded operators on $\mathcal{H}_{\text{ext}} \otimes \mathcal{H}_F$, smooth in the slow variables $x \in M$. We define the total Hamiltonian on a dense domain by

$$H = H_{\text{ext}} \otimes I + I \otimes \omega_* L_F^\beta + V(x), \quad 0 < \beta \leq 1,$$

where L_F^β is defined by the spectral theorem and ω_* is a characteristic inner-time scale.

The exponent β allows anomalous inner-time scaling. In the smooth limit one may take $\beta = 1$. In fractal cases $\beta \neq 1$ can encode nonstandard spectral flow without altering the operator-theoretic definition of the dynamics.

Proposition 3.1 (Well-posedness of dual-time dynamics). Suppose H_{ext} is self-adjoint, L_F is nonnegative and self-adjoint, and $V(x)$ is symmetric and relatively bounded with respect to $H_{\text{ext}} \otimes I + I \otimes \omega_* L_F^\beta$ with relative bound strictly less than 1, uniformly on compact sets in x . Then H is self-adjoint on the domain of the free operator and generates a strongly continuous unitary group e^{-itH} on $\mathcal{H}$.

Proof. This is a direct application of the spectral theorem for the fractional power L_F^β and the Kato-Rellich theorem for relatively bounded perturbations. $\square$

Thus the full DoT evolution is mathematically standard at the Hilbert-space level.

3.2. Projected dynamics and observable unitarity

Define the projected Hamiltonian

$$H_A = (I \otimes P_A)H(I \otimes P_A)$$

on the projected Hilbert space

$$\mathcal{H}_A = (I \otimes P_A)\mathcal{H}.$$

If $[P_A, H] = 0$, the projected dynamics are exactly unitary on $\mathcal{H}_A$. If the commutator is small in norm, then the projected dynamics are adiabatically unitary up to controlled leakage errors.

Proposition 3.2 (No extra propagating time mode in the observable sector). Under the hypotheses of Proposition 3.1, the observable dynamics on $\mathcal{H}_\Lambda$ are generated by the outer time parameter t alone. The inner-time sector contributes only through the operator L_F and its spectral projectors; it does not define an independent observable timelike propagation parameter.

Proof. The Schrödinger evolution is parameterized by t . The inner-time variable is encoded in the operator L_F acting on the internal factor $\mathcal{H}_F$, and after projection all observables lie in the algebra generated by H_Λ and $\mathcal{A}_{\text{obs},\Lambda}$. There is therefore no second observable time parameter in the projected theory. $\square$

This proposition is the precise technical statement behind the phrase “inner time is generative, not directly observable.”

3.3. Fast-slow splitting

The gauge theory will emerge from a standard fast-slow decomposition. Fix $x \in M$ and define the fast inner-time operator

$$K(x) = \omega_* L_F^\beta + V_F(x),$$

where $V_F(x)$ acts only on $\mathcal{H}_F$ (or on the fast tensor factor after a chosen decomposition). The slow variables x are treated adiabatically. For each x , the spectrum of $K(x)$ is assumed to decompose into isolated clusters.

Assumption 3.3 (Protected cluster hypothesis). There exist three smooth spectral subbundles over M ,

$$E_1, E_2, E_3 \rightarrow M,$$

associated with cluster projectors $P_1(x), P_2(x), P_3(x)$ of constant ranks

$$\text{rank} E_1 = 1, \quad \text{rank} E_2 = 2, \quad \text{rank} E_3 = 3,$$

separated from the rest of the spectrum by positive gaps uniform on compact subsets of M .

The subscript denotes rank, not electric charge or color. These bundles will become the Abelian, weak, and color sectors after the structure-group reductions below.

4. Emergent gauge structure from inner-time holonomy

4.1. The universal adiabatic connection

Let $E \rightarrow M$ be any smooth eigenspace bundle defined by a cluster projector $P(x)$ of constant rank m . Choose a local orthonormal frame $\{|u_a(x)\rangle\}_{a=1}^m$ of E . The adiabatic connection one-form is

$$A_\mu(x)_{ab} = i \langle u_a(x), \partial_\mu u_b(x) \rangle.$$

Equivalently, in gauge-covariant form,

$$A_\mu = i P \partial_\mu P|_E.$$

Its curvature is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The holonomy along a loop $\gamma \subset M$ is the path-ordered exponential

$$\text{Hol}_\gamma(A) = \mathcal{P}\exp\left(i \int_\gamma A\right).$$

The following theorem is standard in geometric quantum mechanics but crucial for DoT.

Theorem 4.1 (Gauge emergence from protected eigenspace bundles). Let $E \rightarrow M$ be a smooth spectral subbundle of rank m associated with an isolated cluster of the fast operator $K(x)$. Under adiabatic transport of the slow variables x , the bundle E carries a natural unitary connection whose holonomy group is a subgroup of $U(m)$. In local frames the connection is given by the Wilczek-Zee one-form above.

Proof. The rank- m eigenspaces form a Hermitian vector bundle over M . Adiabatic parallel transport preserves the inner product on each fiber, hence the structure group is unitary. In local orthonormal frames the induced connection is exactly the Berry-Simon-Wilczek-Zee connection. See [8,9]. ▫

Theorem 4.1 gives the central mechanism: gauge fields are holonomies of protected inner-time eigenspace bundles.

4.2. The Abelian sector

If $m = 1$, the bundle is a complex line bundle $L \rightarrow M$.

Corollary 4.2 (Rank-1 sector). If E_1 has rank 1, its adiabatic connection has structure group $U(1)$.

Proof. The unitary group of a one-dimensional Hermitian fiber is $U(1)$. ▫

In the present framework this rank-1 sector should be identified with the Abelian hypercharge factor $U(1)_Y$. Electromagnetism itself arises only after electroweak symmetry breaking, which is not derived in this paper.

Charge quantization in the Abelian sector is topological: isomorphism classes of Hermitian line bundles are classified by the first Chern class $c_1(L) \in H^2(M, \mathbb{Z})$. Tensor powers $L^{\otimes q}$ correspond to integer charges q .

4.3. The weak sector from a quaternionic doublet

The naive structure group of a rank-2 Hermitian bundle is $U(2)$. To obtain $SU(2)$ one needs a reduction.

Assumption 4.3 (Quaternionic structure on the weak bundle). The rank-2 bundle E_2 is equipped with an antiunitary bundle map

$$J: E_2 \rightarrow E_2, \quad J^2 = -I,$$

and the adiabatic transport preserves J .

This is the natural finite-dimensional signature of a quaternionic or symplectic structure.

Corollary 4.4 (Rank-2 quaternionic sector). Under Assumption 4.3, the structure group of E_2 reduces from $U(2)$ to $Sp(1) \cong SU(2)$.

Proof. The unitary automorphisms of a two-dimensional complex Hermitian space that commute with a quaternionic structure J form the compact symplectic group $Sp(1)$, which is canonically isomorphic to $SU(2)$. Since adiabatic transport preserves J , the holonomy lies in this subgroup. ▀

This is the DoT origin of the weak isospin factor.

4.4. The color sector from an oriented triplet

A rank-3 Hermitian bundle has natural structure group $U(3)$. To isolate $SU(3)$ one imposes preservation of complex volume.

Assumption 4.5 (Complex orientation on the color bundle). The rank-3 bundle E_3 carries a nowhere-vanishing complex volume form

$$\Omega \in \Gamma(\Lambda^3 E_3^*)$$

that is preserved by adiabatic transport.

Corollary 4.6 (Rank-3 oriented sector). Under Assumption 4.5, the structure group of E_3 reduces from $U(3)$ to $SU(3)$.

Proof. The subgroup of $U(3)$ preserving a nonzero complex volume form is exactly $SU(3)$, equivalently the determinant-one unitary group. Since the adiabatic transport preserves Ω , the holonomy lies in $SU(3)$. ▀

This is the DoT origin of the color gauge group.

4.5. The minimal DoT gauge theorem

We can now state the main result.

Theorem 4.7 (Minimal DoT gauge sector). Suppose the dual-time system satisfies Assumptions 2.2, 3.3, 4.3, and 4.5. Then the projected observable theory on M carries a natural gauge connection with structure group

$$G_{\text{DoT}} = U(1) \times SU(2) \times SU(3).$$

More precisely, the projected fast eigenspace bundles E_1 , E_2 , and E_3 give rise respectively to a Hermitian line bundle, a quaternionic rank-2 bundle, and a complex-oriented rank-3 bundle, whose adiabatic holonomies produce the three gauge factors.

Proof. By Theorem 4.1, each protected eigenspace bundle carries a unitary adiabatic connection. Corollary 4.2 gives a $U(1)$ factor for E_1 . Corollary 4.4 reduces the rank-2 unitary factor to $SU(2)$. Corollary 4.6 reduces the rank-3 unitary factor to $SU(3)$. Since the three clusters are spectrally isolated, their bundle connections sum as a direct product. ▀

This theorem is the rigorous core of the present manuscript. It turns the original gauge-group claim into a mathematically explicit emergence result.

4.6. What has and has not been proved

The theorem proves an emergence mechanism for the gauge group. It does not yet prove that the observed matter multiplets are exactly realized, that the weak bundle is chiral in precisely the Standard

Model way, that the Higgs field and electroweak symmetry breaking are generated automatically, or that the observed couplings are fixed numerically with no renormalization input. It also does not prove a simple-group grand unification of the Standard Model interactions. The unification established here is weaker and more geometric: a common origin in projected dual-time structure. These are next-step problems, not present accomplishments. A publishable theory must keep that distinction explicit.

5. Effective action and coupling functionals

5.1. Effective action after projection

Let ψ denote a projected matter field taking values in the direct sum bundle

$$E = E_1 \oplus E_2 \oplus E_3.$$

The adiabatic covariant derivative is

$$\nabla_\mu = \partial_\mu + A_\mu^{(1)} \oplus A_\mu^{(2)} \oplus A_\mu^{(3)}.$$

To leading order in slow derivatives, the projected effective action has the form

$$S_{\text{eff}} = \int_{\mathbb{R}_t \times M} d^4x \sqrt{|g|} \left[\bar{\psi} (i\gamma^\mu \nabla_\mu - m) \psi - \sum_{a=1}^3 \frac{1}{4g_a^2(\mu)} \text{tr} (F_{\mu\nu}^{(a)} F_{(a)}^{\mu\nu}) + \dots \right],$$

where $a = 1, 2, 3$ labels the $U(1)$, $SU(2)$, and $SU(3)$ sectors. The omitted terms include scalar fields, higher-derivative operators, and non-adiabatic corrections.

In this paper the effective action is derived only to the extent needed to identify the gauge kinetic terms and their dependence on inner-time spectral data. A full Standard Model reconstruction would require the fermionic representation content and symmetry-breaking sector, which are left open.

5.2. Renormalized coupling functionals

The original draft attempted to compute the couplings directly as exact topological numbers. That is too strong. In a rigorous treatment the coupling constants are renormalized coefficients determined by spectral functionals of the inner-time operator together with matching conditions at a reference scale.

Let f be a positive cutoff profile with Laplace transform

$$f(z) = \int_0^\infty \tilde{f}(s) e^{-sz} ds.$$

For each protected sector define the renormalized inverse coupling functional

$$\mathcal{G}_a(\mu) = \kappa_a \text{Tr} \left(P_a f \left(\frac{L_F}{\mu^2} \right) \right),$$

where P_a projects onto the corresponding fast eigenspace bundle and κ_a is a normalization fixed at a reference scale μ_0 . The physical coupling is then defined by

$$g_a^{-2}(\mu) = Z_a(\mu_0) + \mathcal{G}_a(\mu) - \mathcal{G}_a(\mu_0) + \delta_a^{\text{SM}}(\mu, \mu_0),$$

where $Z_a(\mu_0)$ is the renormalization constant fixed by measurement at μ_0 , and δ_a^{SM} denotes the ordinary four-dimensional running from the projected field content.

This formula has two advantages.

First, it is mathematically meaningful even on singular inner-time spaces because L_F is self-adjoint and the functional calculus is well defined.

Second, it avoids numerology. The theory does not pretend to predict the measured values of the couplings from an unregularized combinatorial formula. Instead it predicts the *form* of the dual-time correction and links all three sectors to the same inner-time spectrum.

5.3. Fractal asymptotics and anomalous running

Suppose now that the heat trace of L_F in sector a has short-time asymptotics

$$\text{Tr}(P_a e^{-sL_F}) \sim s^{-d_s/2} \left(c_{a,0} + \sum_{n \geq 1} c_{a,n} s^{\eta_n} + \sum_{m \geq 1} d_{a,m} \cos(\omega_m \log s + \phi_m) \right)$$

as $s \downarrow 0$, where d_s is the spectral dimension. The oscillatory terms occur in exactly self-similar situations and correspond to complex dimensions of the spectral zeta function [5,11].

Substituting this asymptotic into the Laplace-transform formula for $\mathcal{G}_a(\mu)$ yields two generic conclusions.

1. The dual-time correction to the running of every gauge coupling is controlled by the same spectral dimension d_s and the same family of scaling exponents η_n .
2. If the inner-time space is exactly self-similar, the gauge couplings acquire small log-periodic modulations at high scales.

These are distinctive signatures of fractal inner time. They replace the unsupported claim that the fine-structure constant can be read off as a single exact topological invariant.

5.4. Electroweak mixing

Once a scalar sector triggers electroweak symmetry breaking, the usual relation

$$\tan \theta_W = \frac{g_1}{g_2}$$

holds in the projected theory. In the DoT framework this angle is therefore determined by the ratio of the two renormalized sector functionals. What is *not* yet shown is that the observed value of θ_W follows from the inner-time geometry without additional matching input. The present framework derives the gauge factors and the functional form of the couplings, but not yet their final measured low-energy values.

6. Gravity as projected geometry

6.1. Metric data from the projected spectral geometry

Gauge structure enters through holonomy of eigenspace bundles. Gravity enters differently: it is encoded in the metric content of the projected observable algebra. When a projected spectral triple

$$(\mathcal{A}_{\text{obs}}, \mathcal{H}_{\text{obs}}, D_{\text{obs}})$$

exists, the Connes distance formula defines a metric on the state space of $\mathcal{A}_{\text{obs}}$ by

$$d(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{A}_{\text{obs}}, \| [D_{\text{obs}}, a] \| \leq 1\}.$$

In smooth sectors this reproduces the geodesic distance of the emergent manifold. Within DoT, the metric is therefore not postulated independently; it is extracted from the projected temporal geometry.

6.2. Spectral action and the smooth-sector limit

When the projected geometry is regular enough for the spectral action expansion to apply, one may define

$$S_{\Lambda}^{\text{spec}} = \text{Tr} \chi \left(\frac{D_{\text{obs}}}{\Lambda} \right) + \langle \Psi, D_{\text{obs}} \Psi \rangle.$$

The Chamseddine-Connes spectral action then yields, in the smooth limit, a bosonic action containing Einstein-Hilbert, Yang-Mills, and scalar terms [4]. This observation motivates the DoT interpretation of gravity: it is not an extra gauge factor parallel to $U(1)$, $SU(2)$, and $SU(3)$, but the metric sector of the same projected spectral geometry.

Proposition 6.1 (Smooth-sector gravity statement). If the projected observable algebra admits a regular four-dimensional spectral triple satisfying the hypotheses of the spectral action expansion, then the bosonic part of the projected DoT action contains Einstein-Hilbert and Yang-Mills terms.

Proof. This is the standard spectral-action result applied to the projected triple. The DoT-specific input is that the triple itself arises by projection from the dual-time arena. ▀

6.3. Fractal corrections and open issues

On a genuinely fractal inner-time space the full local spectral-action expansion is not yet available in the same generality as in the smooth case. The expected correction terms are controlled by the fractal heat-kernel asymptotics and may include anomalous powers and log-periodic oscillations. At present this should be stated as a conjectural extension, not as a proved Einstein equation on a fractal temporal background.

Thus gravity is incorporated at two levels: rigorously in the smooth projected sector via standard spectral geometry, and conjecturally in the fully fractal regime where only asymptotic control is presently available.

6.4. Geometric hierarchy and qualified unification of forces

The results of Sections 4 through 6 permit a sharpened statement of what unification means in the DoT framework. The relevant unity is not obtained by postulating a single simple gauge group on an

already given spacetime. Rather, it is obtained by showing that coupling data, gauge holonomy, and gravitational metric structure are all projected from the same dual-time geometry, albeit through different mathematical channels.

Proposition 6.2 (Qualified geometric unification of forces). Suppose the hypotheses of Theorem 4.7 hold and, in addition, the projected observable algebra admits a regular four-dimensional spectral triple satisfying the assumptions of Proposition 6.1. Then the bosonic interactions of the projected DoT theory are unified at the level of geometric origin: the renormalized gauge couplings are determined by spectral functionals of the common inner-time operator, the gauge fields of $U(1) \times SU(2) \times SU(3)$ arise as adiabatic holonomies of protected eigenspace bundles of the same inner-time geometry, and gravity arises from the metric sector of the projected spectral triple induced by the same projection. This does not yet constitute grand unification in the sense of a single simple connection or exact coupling equality, but it does provide a single geometric source for the bosonic sectors.

Proof. The common spectral origin of the coupling functionals was established in Section 5.2, where all sectors are defined from the same inner-time operator and the same projection procedure. The emergence of the three gauge factors from protected eigenspace bundles was proved in Theorem 4.7. The emergence of gravity from the metric sector of the projected spectral geometry was established in Proposition 6.1 together with the discussion of Section 6.1. Since these structures are all extracted from the same projected dual-time data, the theory is unified at the level of geometric origin, even though it is not yet unified in the stronger simple-group sense. $\square$

Remark 6.3 (Dimensional hierarchy). This proposition also clarifies the heuristic language of $0D+T$, $1D+T$, $2D+T$, and $3D+T$. These labels should not be read as separate observable spacetimes. In DoT they denote layers of projected temporal geometry. At the $0D+T$ level one has spectral data and scalar invariants; this is the level at which coupling functionals are defined. At the $1D+T$ level one has phase-line geometry, giving the Abelian $U(1)$ sector. At the $2D+T$ level one reaches the first genuinely non-Abelian bundle geometry, realized minimally by the quaternionic doublet that yields $SU(2)$. The $3D+T$ label then plays a dual role: internally it appears as the oriented triplet responsible for the color sector $SU(3)$, while externally it appears through the rank-3 observable cotangent structure and the emergent three-dimensional spatial manifold whose metric sector yields gravity. What is unified is therefore not a stack of extra macroscopic dimensions, but a hierarchy of spectral, bundle, and metric geometries generated from one dual-time source.

In this precise sense the DoT program supports a qualified unification of forces. Gauge interactions are holonomies of protected inner-time bundles, gravity is the metric content of the projected observable geometry, and the coupling hierarchy is encoded in the same inner-time spectrum. The framework therefore unifies origin without yet claiming full dynamical completion of the Standard Model or conventional grand unification.

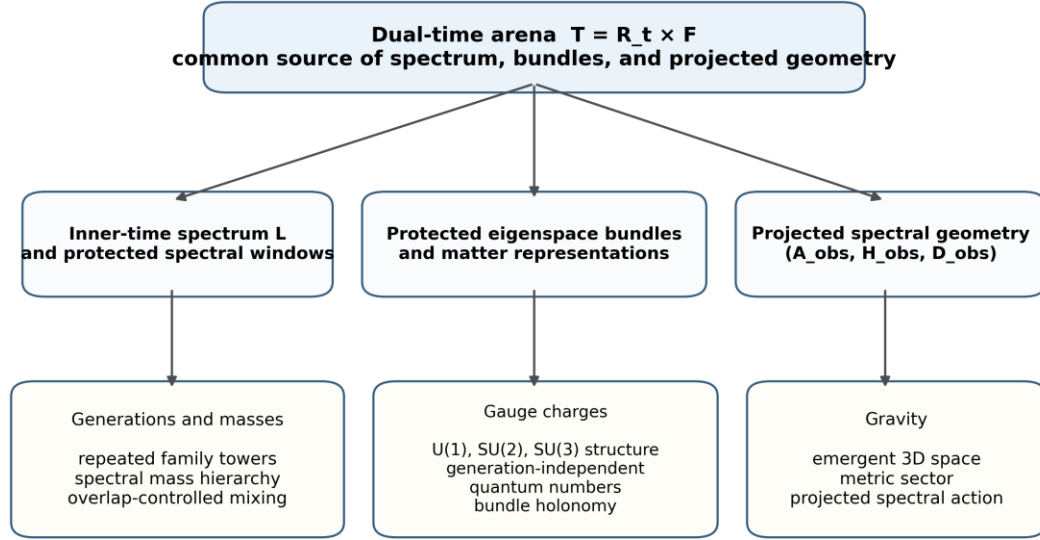
6.5. Generations, charges, and masses from spectral replication

The geometric hierarchy described in Section 6.4 also suggests a controlled extension from bosonic unification to the replicated matter content of the Standard Model. The key distinction is between representation type and spectral level. Gauge quantum numbers are fixed by the protected bundle type on which a field lives, whereas the generation label records which protected spectral window of the

inner-time operator supports that field. In this picture the family structure of the Standard Model is not a second copy of gauge symmetry, but a spectral replication of the same gauge representation.

observable matter tower = first copy $\oplus$ second copy $\oplus \dots \oplus$ Nth copy, with every copy carrying the same $U(1) \times SU(2) \times SU(3)$ representation.

Figure 1 visualizes the resulting separation of roles: spectrum controls family replication and masses, bundle holonomy controls charges, and projected metric geometry controls gravity.



*Unified origin in DoT: spectrum controls family replication and masses,
bundle holonomy controls charges, and projected metric geometry controls gravity.*

Figure 1. Common geometric source of generations, charges, and gravity in the DoT framework.

Hypothesis 6.4 (Spectral replication of matter). For each admissible matter representation of $U(1) \times SU(2) \times SU(3)$, there exist finitely many protected matter bundles associated with disjoint spectral windows of the inner-time operator. All such bundles carry the same gauge representation, but their spectral centers and overlap data with the symmetry-breaking sector may differ.

Proposition 6.5 (Generation independence of charges). Under Hypothesis 6.4, gauge charges are generation-independent. After electroweak symmetry breaking the familiar electric charges are likewise generation-independent.

Proof. By construction the generation index labels only the spectral window, not a change of bundle type or gauge representation. The $U(1)$, $SU(2)$, and $SU(3)$ quantum numbers are therefore fixed across the replicated family. After symmetry breaking the electric charge operator is built from the same Abelian and weak representation data, so the resulting electric charges also remain the same across the family. $\square$

The separation between generation-independent and generation-dependent quantities is summarized in Table 1.

Table 1. Geometric control of generation-independent and generation-dependent observables in DoT.

Feature	Controlling geometric datum	Generation dependence	Status in DoT
Gauge group	Protected bundles E1, E2, E3 and their structure-group reductions	Common to all generations	Proved
Gauge / electric charges	Representation type on replicated matter bundles	Independent of n	Controlled mechanism
Number of generations	Number of protected matter windows below the cutoff	Depends on spectrum	Open
Fermion masses	Spectral centers of the replicated windows and overlap with the symmetry-breaking sector	Depends on n	Qualitative mechanism
CKM / PMNS mixing	Off-diagonal overlaps between replicated windows	Depends on n	Open
Gravity	Projected metric geometry and spectral action	Universal	Proved in smooth sector

Once a scalar sector of the type anticipated in Section 5.4 is included, the effective Yukawa couplings may depend on spectral position. A convenient parameterization is

$$\text{effective Yukawa} = \text{base Yukawa} \times \text{spectral-response factor}, \quad \text{mass} = (v / \sqrt{2}) \times \text{effective Yukawa}.$$

If the response functional is monotone on the first observable windows, then one obtains an ordered mass tower across the first three replicas without changing the underlying charges. The framework therefore relocates mass hierarchy from arbitrary generation labels to the spectral organization of inner time. This remains a mechanism rather than a completed derivation, because the symmetry-breaking sector and the precise overlap functionals are not yet fixed by the present axioms.

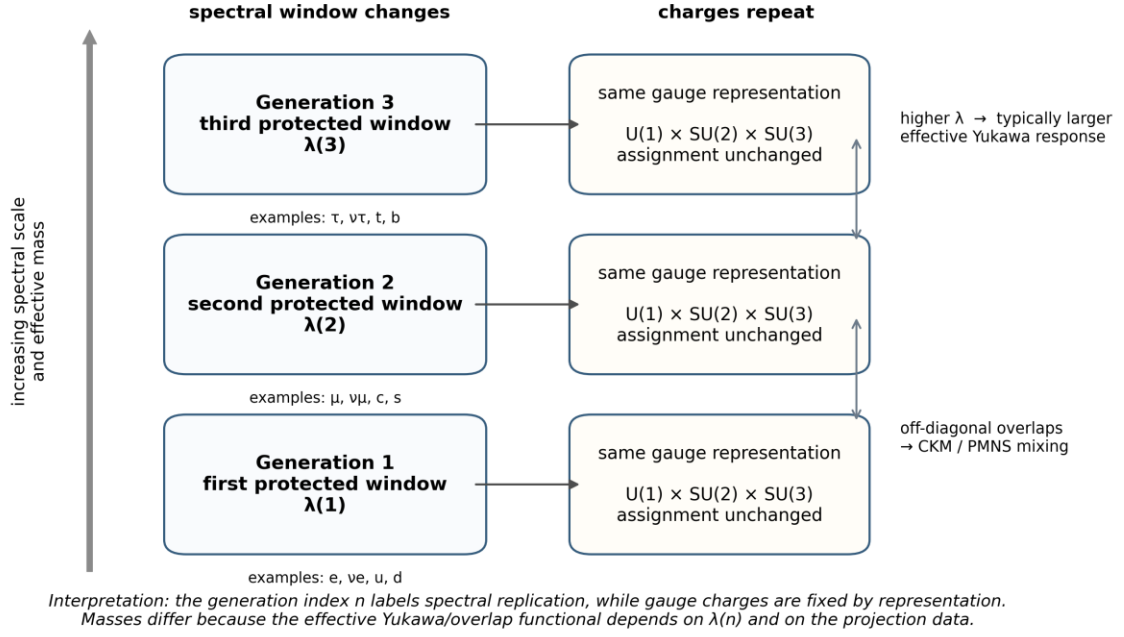


Figure 2. Spectral replication of a fixed matter representation: charges repeat because the gauge representation is unchanged, while masses and mixing depend on spectral level and overlap.

Table 2 gives an illustrative reading of the observed Standard Model family pattern in this language [13].

Table 2. Illustrative reading of the Standard Model family pattern as spectral replication.

Tower	States	Common charge pattern	DoT reading
Charged leptons	e, μ, τ	$Q = -1$, color singlet	Same lepton representation on three spectral windows; spectral level orders the masses
Neutrinos	electron, muon, tau neutrinos	$Q = 0$, color singlet	Same neutral-lepton representation; masses and mixing are especially sensitive to overlap data
Up-type quarks	u, c, t	$Q = +2/3$, color triplet	Same quark representation on three spectral windows; spectral level orders the masses
Down-type quarks	d, s, b	$Q = -1/3$, color triplet	Same quark representation on three spectral windows; spectral level orders the masses

Remark 6.6 (Why three generations?). In the present formulation the empirical number three is interpreted as the number of replicated matter windows that remain protected and below the observable

cutoff for a given admissible matter representation. This gives the theory a natural location for the observed three-family structure, but it does not yet prove that only three such windows can occur.

Nontrivial CKM and PMNS matrices can arise when the effective Yukawa map or the projection operator is not diagonal in the spectral-replication basis. In that case the mixing amplitudes are overlap integrals between neighboring replicated windows. The qualitative prediction is therefore sharp even where the numerics are not: charges are fixed by bundle type, masses by spectral position, and flavor mixing by off-diagonal overlap. In this way the same geometric hierarchy that unifies the bosonic sectors also provides a principled place for generations, charge replication, and mass hierarchy.

7. Falsifiable consequences

A mathematically responsible speculative theory should make predictions whose form is fixed even when all numerical coefficients are not yet known. The DoT framework yields four such signatures.

7.1. Correlated high-scale corrections to gauge running

Because all three coupling functionals are built from the same inner-time operator L_F , their beyond-Standard-Model corrections are not arbitrary. They share the same spectral exponents. In particular, once the scale Λ_* associated with the first nontrivial inner-time correction is crossed, the couplings should deviate from ordinary four-dimensional running in a correlated way rather than independently.

A minimal phenomenological parameterization is

$$\mu \frac{d}{d\mu} g_a^{-2}(\mu) = \frac{b_a}{8\pi^2} + C_a \Phi\left(\frac{\mu}{\Lambda_*}\right),$$

where b_a are the usual beta-function coefficients and the same profile Φ appears in all sectors, with sector dependence only in the amplitudes C_a .

7.2. Possible log-periodic modulation

If F is exactly or approximately self-similar, then the spectral zeta function can develop complex poles, implying oscillatory corrections periodic in $\log \mu$. The resulting coupling modulation is of the schematic form

$$\delta g_a^{-2}(\mu) \sim A_a \cos(\omega \log(\mu/\Lambda_*) + \phi_a).$$

Such a signal would be exotic but highly characteristic: it would be difficult to mimic by ordinary threshold corrections.

7.3. Low-energy decoupling of the hidden temporal sector

Below the projection scale, the hidden temporal sector decouples except through renormalized constants. Hence the framework predicts no directly observable second timelike coordinate, no extra low-energy causal structure, and no macroscopic violation of single-time evolution. This is an internal consistency requirement as much as an empirical statement.

7.4. Generation replication and family-scaling relations

If the matter sector is realized by spectral replication, then masses within a fixed charge and color tower are not arbitrary independent inputs. They should be organized by the spectral gaps between successive protected windows and by the same overlap data that generate mixing. A minimal expectation is therefore that mass ratios within a fixed tower are constrained functions of the underlying spectral separations rather than unrelated Yukawa choices.

In nearly self-similar or fractal regimes the same spectral exponents that modify gauge running may also induce approximate power-law or log-periodic regularity across family towers. Likewise, CKM and PMNS entries should correlate with localization properties of neighboring replicated windows rather than behave as statistically independent parameters. These are not yet numerical predictions, but they are falsifiable structural constraints on any completed DoT matter sector.

8. Scope, limitations, and route to a full Standard Model

The present paper is intentionally narrower than the original draft. It formulates DoT on an explicit dual-time Hilbert space with a self-adjoint inner-time generator, defines the observable algebra by spectral projection, proves that protected inner-time eigenspace bundles yield gauge holonomy, proves that the minimal multiplicity pattern (1, 2, 3) with natural structure reductions gives $U(1) \times SU(2) \times SU(3)$, defines renormalized coupling functionals controlled by inner-time spectral data, explains how gravity enters as the projected metric sector, shows how these ingredients amount to a qualified geometric unification of the bosonic forces, and formulates a controlled matter-sector hypothesis in which generations arise as spectral replicas with common gauge charges and generation-dependent masses.

Equally important is what the paper does not yet accomplish: it does not derive the full chiral matter content of the Standard Model, it does not yet derive the Higgs field or electroweak symmetry breaking from first principles, it does not compute the observed couplings exactly and without matching input, it does not prove that exactly three generations must occur for every admissible matter representation, it does not compute CKM or PMNS matrices numerically, it does not generate the measured Yukawa hierarchies from first principles, and it does not yet classify all inner-time geometries whose low-energy sector reproduces the Standard Model. These limitations are not defects of presentation; they are the actual research frontier. A top-tier submission must state them plainly.

The route forward is clear. First, one must classify admissible inner-time geometries with protected cluster pattern (1, 2, 3). Second, one must construct a fermionic representation scheme that reproduces Standard Model chirality and determine how many replicated matter windows survive below the observable cutoff. Third, one must identify the scalar sector responsible for electroweak breaking and compute the induced overlap and Yukawa functionals. Fourth, one must verify anomaly cancellation and derive the CKM and PMNS structures. Fifth, one must turn the spectral coupling and mass functionals into precise phenomenology with explicit thresholds and bounds.

9. Conclusion

The original motivation of the Duality of Time program was bold: space and physical law might emerge from a deeper temporal structure. The present paper shows that part of that ambition can be made mathematically precise.

Starting from a dual-time arena $T = R_t \times F$, with inner-time geometry encoded by a Dirichlet form and, when available, a spectral triple, we defined the projected observable sector and proved that adiabatic transport of protected inner-time eigenspace bundles naturally yields gauge connections. Under a minimal and explicit multiplicity pattern, together with a quaternionic reduction in rank 2 and an oriented reduction in rank 3, the resulting gauge group is exactly $U(1) \times SU(2) \times SU(3)$.

This is not yet a complete unification of all known physics. But it does provide a rigorous DoT-based unification of the bosonic sectors at the level of geometric origin: coupling data arise from the common inner-time spectrum, gauge structure arises from bundle holonomy, and gravity arises from the metric content of the same projected geometry. The same hierarchy also suggests a controlled matter-sector extension in which repeated spectral windows carry identical gauge charges across generations while their spectral position and overlap data control mass hierarchy and mixing. In that sense it replaces informal analogies with operator theory, vector bundles, holonomy, and spectral analysis, and transforms the original proposal from a conceptual manifesto into a mathematical research program.

The next stage is now well defined: build the matter sector, prove or refute the three-generation cutoff, derive symmetry breaking, compute mixing, and confront the spectral coupling and mass corrections with experiment. If those steps succeed, Duality of Time will move from an interpretive framework to a genuine competitor in fundamental theory. If they fail, the current formulation will still have identified - with mathematical clarity - exactly where the idea succeeds and where it breaks.

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