

Gauge Structure from Temporal Completion in SMM/DTT

Exact singlet-loop results and conditional operational reconstruction

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Abstract

We prove three model-level results for a four-position mobile carrier bound to a stationary dual by a rank-one singlet penalty. First, an exact holonomy-dependent resolvent determines its ground energy for every positive singlet penalty Δ and real hopping. The ground state is unique; for carrier rank $m \geq 2$ the gap is exact, with sharp uniform lower bound $\Delta/(4m)$. Second, the exact identity Hessian contains $\text{Tr } X^2$ but no independent $(\text{Tr } X)^2$ term. For independent rank-three and rank-two auxiliaries this gives $\hat{\kappa}_1 = 12\hat{\kappa}_3 + 18\hat{\kappa}_2$ at every hopping order in the frozen-link model; equal gaps and equal nonzero hoppings give $\hat{\kappa}_2/\hat{\kappa}_3 = 3/2$. Third, exact unitary compact links with bounded electric dispersion on a finite cell complex admit a controlled effective Hamiltonian with an explicit sixth-order norm remainder and a dressed evolution estimate. Representation compression requires additional boundary terms.

These results require specified Hamiltonians and do not assume the Single Monad Model (SMM) or Duality of Time Theory (DTT). A separate conditional reconstruction interprets generative histories and completed records: future interference constrains retained transport, independently unframed fibers require relational comparison, and stipulated carrier and determinant data select $S(U(3) \times U(2))$. The record update, flat-source counterexamples, rotor circulation obstruction, and one-generation exterior module expose the additional assumptions. Spatial dimension, canonical anticommutation relations, Weyl chirality, electric coefficients, and physical matching remain unresolved. Neither a first-principles Standard Model derivation nor empirical identification of SMM/DTT is claimed.

Keywords: gauge reconstruction; singlet-loop model; effective Hamiltonians; quantum records; gauge holonomy; foundations of physics.

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1 Results, scope, and antecedents

The manuscript separates three kinds of statements: *proved model results* for specified finite or bounded Hamiltonians; *conditional implications* for temporal records satisfying declared operational principles; and *additional assumptions* that select source classes, interfaces, preparations, matter, and continuum interpretations. Their logical roles are not interchangeable. The mathematical core in Part I is independent of the SMM/DTT interpretation in Part II.

1.1 Summary of new results

1. **Exact singlet-loop resolvent and gap.** Theorem 2.1 determines the holonomy-dependent ground branch at every real hopping and positive penalty, proves uniqueness, and gives an exact gap for $m \geq 2$ with sharp uniform bound $\Delta/(4m)$.
2. **Exact identity Hessian.** Theorem 2.3 calculates the full static coefficient and excludes an independent trace-square quadratic invariant. Corollary 2.4 gives the frozen-link relation $\hat{\kappa}_1 = 12\hat{\kappa}_3 + 18\hat{\kappa}_2$; equal block parameters give an exact ratio of $3/2$.
3. **Controlled finite-volume dynamics.** Theorems 2.5 and 2.6 calculate the fourth-order interaction and bound the sixth-order remainder for bounded electric dispersion, with an explicit dressed evolution estimate. The finite Peter–Weyl version retains cutoff-boundary polynomials.

Here *new* identifies the model-specific contributions emphasized in this paper, not an assertion of historical priority for the underlying resolvent or perturbation methods.

What is not claimed. The paper does not derive the specified singlet Hamiltonian, quantum links, record graph, determinant reference, CAR, or physical Weyl chirality from bare SMM/DTT. It does not determine $\beta_3, \beta_2, \beta_1$, observed gauge couplings, masses, or three generations. It does not establish a regulator-uniform thermodynamic or continuum limit, Lorentz symmetry, confinement, or a four-dimensional Yang–Mills mass gap. It supplies no proof that nature realizes SMM/DTT and no unconditional Standard Model derivation.

1.2 Reading order and independent branches

Part I states its Hilbert spaces, Hamiltonians, group, normalization, and regulator without invoking the temporal source. Part II develops operational necessity, source and record assumptions, no-go results, the joint update, and conditional one-generation matter. Part III consolidates claim status and open proof obligations. The dependency ledger in Section 4.2 lists every source, R, A, M, and D label. The reproducibility checklist in Appendix A.1 distinguishes numerical checks from analytical proof obligations.

1.3 Novelty and antecedents

The mathematical comparison must concern hypotheses and conclusions, not the temporal terminology. Transport reconstruction, virtual plaquette generation, unitary low-energy reduction, and exterior-algebra packaging of matter all have direct antecedents. The following comparison locates the results proved in this paper inside those established constructions.

Table 1: Antecedents, reused constructions, and model-specific contributions.

Antecedent	Hypotheses and established conclusion	Present specialization and boundary
Transport reconstruction [26]	Smooth transport functors with local trivializations and smooth descent data on the thin path groupoid correspond to principal bundles with connection (Theorem 5.4), and to Hermitian vector bundles with unitary connection (Theorem 5.8).	The future-test argument constrains which history distinctions a sufficient record may erase. It supplies an operational premise for retaining transport; the subsequent connection correspondence is reused, not rediscovered. The group, smooth base, and descent conditions remain inputs.
Generated plaquettes [31]	Gauge-invariant matter–link hopping, penalized virtual auxiliary motion, and specified preparations generate trace plaquettes at fourth order. Section VI also calculates electric renormalization; Section VII discusses nonunitary finite-link realizations.	The present four-position mobile carrier and stationary dual use a rank-one bound singlet. Their exact resolvent determines the entire static ground branch and its identity Hessian. The general virtual loop mechanism and trace contraction are antecedents; the coefficient and exact Hessian constraint concern this particular binding Hamiltonian.
Hamiltonian lattice gauge theory [12]	Compact link variables with conjugate electric generators give electric Casimirs, plaquette magnetic terms, and local Gauss generators.	The generated potential is inserted into this established compact-link setting. The electric coefficients and neutral sector are specified separately. A static magnetic relation does not select physical couplings or a common light cone.
Controlled elimination [35]	A spectrally separated sector and bounded perturbation admit exact direct rotation and a convergent Schrieffer–Wolff expansion. Local spin hypotheses additionally yield linked-cluster and extensive ground-energy estimates.	The finite-volume result below proves a quantitative error for the stated model, reduction, electric scale, and representation treatment. It is a model application with explicit constants, not a new general perturbation method or an imported thermodynamic theorem.

Compact rotor plaquettes [33, 30]	Auxiliary rotor transfer generates ring exchange; the circuit construction of [33], Eq. (9), gives the square coefficient $-5\lambda^4 \cos \theta / (16\mu^3)$.	Matching $\Delta = 2\mu$ reuses that square coefficient. The simple-cycle denominator count and circulation obstruction are analyzed here; no new square ring-exchange mechanism is claimed.
Exterior matter [1]	The splitting of a five-dimensional unitary carrier and its exterior/spin representations encode the familiar $SU(5)$ and $\text{Spin}(10)$ generation structure, including the Standard-Model subgroup and charge relations.	Reusing the proposed temporal $3 + 2$ carrier under M1–M4 calculates the same representation. Neither the exterior decomposition nor its anomaly cancellation is new; CAR, parity selection, and physical Weyl chirality remain additional.
Record retention and reference resources	Standard-form expectations, Lindblad channels [24, 25], and quantum reference frames [4] provide established tools.	The explicit joint update exposes its retained algebra and resource budget. The joint-right preservation obstruction follows from the multiplicative domain argument; it is not a novel general theorem about completely positive maps. Its role is to rule out an incompatible proposed record-erasure mechanism.

The quantitative addition is the exact static singlet-loop response: the physical secular branch, a proved isolation bound, the full small-curvature coefficient, and a two-block Hessian relation that survives all orders of *the same frozen-link auxiliary elimination*. The finite-volume reduction then states what remains controlled when electric motion is restored. The joint-update construction links generation, completion, retained comparisons, and explicitly accounted references; where it cannot select a desired structure, the obstruction or extra preparation is stated. These are reproducible mathematical results for specified models. They do not establish historical priority for every special-case identity, uniquely derive their microscopic inputs, or identify nature with SMM/DTT.

Other retained results have independent roles. The rotor calculation extends the explicit simple-cycle denominator count, while its circulation symmetry gives a normalizability obstruction. The square coefficient has a direct recent antecedent: the circuit rotor model of [33], Eq. (9), gives $-5\lambda^4 \cos \theta / (16\mu^3)$ when its matter charging coefficient is denoted here by μ ; setting $\Delta = 2\mu$ agrees with our convention. No new square ring-exchange mechanism is claimed. The dimensional countermodels, flat source realization, twirl/sewing distinction, and color/family obstruction delimit the reconstruction rather than count as phenomenological evidence.

Part I

Exact mathematical model results

2 Exact singlet-loop stiffness and controlled finite-volume dynamics

This section is a self-contained mathematical study of a specified singlet-binding Hamiltonian. All carriers are ordinary complex Hermitian spaces, and all penalties, hoppings, and link variables are inputs. Neither a temporal source nor record completion is needed for its proofs. The general auxiliary virtual-loop method is established [31, 32]; the model-specific results are the exact secular branch, its isolation bound, the identity Hessian, and the controlled finite-volume implementation.

For the two-block application choose, as a mathematical input,

$$G = G_{\text{src}} := \{(g_3, g_2) \in U(3) \times U(2) : \det g_3 \det g_2 = 1\}.$$

Use its defining unitary representations on $E_m = \mathbb{C}^m$, $m = 3, 2$, and their duals. The single-block static proofs also apply to arbitrary $m \geq 1$ and $W \in U(m)$. The primitive integral central generator in the two-block specialization is $i \text{diag}(-2I_3, 3I_2)$; the later conditional reconstruction explains when this chosen group is a record stabilizer (Section 11). This interpretation is not an assumption in the proofs below. The configuration group is compact, with normalized Haar measure; electric Casimirs in Section 2.3 use its simple and central factors.

2.1 The retained link variable and the singlet auxiliary space

Fix an oriented four-edge cycle with vertices 0, 1, 2, 3. On each edge place $g_e \in G_{\text{src}}$, with defining block representations $U_{e,m} \in U(m)$ for $m = 3, 2$. Their product determinants are one on each edge. These are variables of the construction; their group is the explicitly chosen compact group above. For one block let

$$\mathcal{K}_m = \bigoplus_{j=0}^3 E_{m,j} \otimes E_{m,0}^*, \quad \dim \mathcal{K}_m = 4m^2. \quad (1)$$

The first factor is a mobile carrier, while its dual reference stays at the root. The normalized identity tensor at vertex zero is

$$|s_m\rangle = \frac{1}{\sqrt{m}} \sum_{a=1}^m |0, a\rangle \otimes |\bar{a}\rangle, \quad P_m = |s_m\rangle\langle s_m|, \quad Q_m = I - P_m.$$

This singlet is basis independent. It supplies a physical local carrier–dual contraction, rather than an averaging over an arbitrary color state after a calculation.

For a directed edge $j \rightarrow j+1$ (indices modulo four), let $T_{j,m}$ move the carrier with matrix $U_{j+1,j;m}$ and act as identity on the dual factor. It vanishes on the other position summands. Define

$$H_m(g) = \Delta_m Q_m + V_m(g), \quad V_m(g) = -t_m \sum_{j=0}^3 (T_{j,m} + T_{j,m}^*), \quad \Delta_m > 0. \quad (2)$$

All unwanted positions and root nonsinglets cost Δ_m . This is a chosen singlet-binding and displacement penalty; it is not identified with the seven-clock frequency. The graph degree is two, so $\|V_m\| \leq 2|t_m|$. The elementary perturbation intervals already isolate the zero-level continuation when $|t_m| < \Delta_m/4$. The exact rank-one argument below removes that small-hopping restriction for the static ground state and supplies a uniform gap. A controlled low-order dynamical approximation still requires small hopping.

Under an active frame change h_j , the auxiliary transformation on the j th summand is $h_j \otimes \bar{h}_0$ and $U_{j+1,j} \mapsto h_{j+1} U_{j+1,j} h_j^{-1}$. The singlet at zero is invariant, and every term in the Hamiltonian transforms covariantly. Thus it defines a gauge-invariant operator when the links are quantized together with these auxiliary charges. The root's position is chosen, but no root color frame is chosen.

2.2 Exact ground band and static stiffness

Let $W_m = U_{0,3;m} U_{3,2;m} U_{2,1;m} U_{1,0;m}$ be forward transport around the cycle. Reversal replaces W_m by W_m^* . In this subsection the links are frozen configuration variables. No quantum electric operator is included in the elimination.

2.2.1 Rank-one resolvent and the physical branch

Suppress the block label and write $\Delta > 0$, $t \in \mathbb{R}$, $m \geq 1$. A vertex frame transformation makes the first three links identity and the last link W . Diagonalize W with eigenvalues $e^{i\theta_a}$. The simultaneous change of mobile and dual root basis preserves s . Let $K = t \sum_j (T_j + T_j^*)$, so that

$$H = \Delta I - (K + \Delta|s\rangle\langle s|).$$

The mobile four-position operator splits into twisted rings. Its eigenvalues, including multiplicities in the mobile space but not the spectator dual space, are

$$\lambda_{ak} = 2t \cos \frac{\theta_a + 2\pi k}{4}, \quad a = 1, \dots, m, \quad k = 0, 1, 2, 3, \quad \lambda_* = \max_{a,k} \lambda_{ak}. \quad (3)$$

Changing the twist convention replaces θ_a by $-\theta_a$ and leaves this multiset unchanged. Each normalized twisted-ring mode has root probability $1/4$; the normalized identity contraction therefore assigns spectral weight $1/(4m)$ to each displayed mode. The free operator on $\mathcal{K}_m$ repeats every mobile eigenvalue m times because it acts trivially on the dual.

Theorem 2.1 (Exact singlet-loop ground band and gap). *For every $\Delta > 0$, every real t , and every $W \in U(m)$, the Hamiltonian (2) has a unique ground state. Its energy is $E_m(W) = \Delta - z(W)$, where $z(W)$ is the unique real solution with $z > \lambda_*$ of*

$$1 = \Delta R_W(z), \quad (4)$$

$$R_W(z) = \frac{1}{4m} \sum_{a=1}^m \sum_{k=0}^3 \frac{1}{z - \lambda_{ak}} = \frac{1}{m} \sum_{a=1}^m \frac{z(z^2 - 2t^2)}{z^4 - 4t^2 z^2 + 2t^4(1 - \cos \theta_a)}. \quad (5)$$

The eigenvector is proportional to $(z - K)^{-1}s$. If $w_* = \|\mathbf{1}_{\{\lambda_*\}}(K)s\|^2$, then the full auxiliary gap satisfies

$$\text{gap } H_m(W) \geq z(W) - \lambda_* \geq \Delta w_* \geq \frac{\Delta}{4m}. \quad (6)$$

For $m \geq 2$, the first inequality is an equality. At $t = 0$ the energy is zero and the gap is Δ . For the physical blocks $m = 2, 3$, the bound $\Delta/(4m)$ is sharp as a uniform lower bound over unrestricted hopping and holonomy.

Proof. For $z > \lambda_*$, the equation $(K + \Delta|s\rangle\langle s|)\psi = z\psi$ gives $\psi = \Delta(z - K)^{-1}s\langle s, \psi \rangle$. A nonzero vector on this branch has nonzero overlap with s ; contraction gives $1 = \Delta R_W(z)$. Conversely that scalar equation constructs a nonzero eigenvector. The spectral weights above give the first expression for R_W . For one ring put

$$D_\theta(z) = \prod_{k=0}^3 (z - \lambda_{ak}) = z^4 - 4t^2 z^2 + 2t^4(1 - \cos \theta).$$

Its root diagonal resolvent is $\frac{1}{4}D'_\theta(z)/D_\theta(z)$, which gives the rational expression. Apparent cancellations at special twists are understood through the spectral-sum expression.

All spectral weights are positive. Hence R_W decreases strictly from infinity at λ_*^+ to zero at infinity. The root is unique and simple. Rank-one eigenvalue interlacing places all other eigenvalues of $K + \Delta|s\rangle\langle s|$ at or below λ_* , giving a unique ground state of H and the first gap bound. For $m \geq 2$ the λ_* eigenspace of K has dimension at least two. A vector in it orthogonal to s remains an eigenvector with eigenvalue λ_* , so the gap is exactly $z - \lambda_*$. For $m = 1$ such a vector need not exist; only the inequality is asserted. The secular equation implies $1 \geq \Delta w_*/(z - \lambda_*)$. For nonzero t , w_* equals the number of maximizing mobile modes divided by $4m$, and is at least $1/(4m)$. For $t = 0$, $w_* = 1$. To see sharpness for $m \geq 2$, choose holonomy angles with a unique maximizing mobile mode and scale $|t|/\Delta \rightarrow \infty$ at fixed angles. All other free eigenvalues separate from the maximum linearly in $|t|$. The secular equation then gives $z - \lambda_* \rightarrow \Delta/(4m)$. $\square$

This gap theorem holds outside the perturbative regime. It does not supply a uniform low-energy expansion after unbounded electric motion is added. The eigenvalue $z > \lambda_*$, rather than an arbitrary root obtained after clearing secular denominators, always defines the physical branch.

At $W = I$, the physical root z_0 satisfies

$$z_0^3 - \Delta z_0^2 - 4t^2 z_0 + 2\Delta t^2 = 0, \quad z_0 > 2|t|, \quad (7)$$

with $z_0 = \Delta$ at $t = 0$. The root is independent of m . For $m = 2, 3$, the identity gap is exactly $z_0 - 2|t| \geq \Delta/4$ when $t \neq 0$. This stronger identity bound is also sharp as $|t|/\Delta \rightarrow \infty$.

Corollary 2.2 (Strict minimum at trivial holonomy). *For $t \neq 0$, $E_m(W) \geq E_m(I)$, with equality precisely when $W = I$.*

Proof. Evaluate the secular function at $z_0 > 2|t|$. Its positive numerator is independent of θ_a , while each denominator is the identity denominator plus $2t^4(1 - \cos \theta_a)$. Thus $R_W(z_0) \leq R_I(z_0) = 1/\Delta$, with equality only if all $e^{i\theta_a} = 1$. Strict monotonicity gives $z(W) \leq z_0$ and the energy claim. $\square$

2.2.2 Exact Hessian, including central directions

Theorem 2.3 (Exact static stiffness and absence of a trace-square term). *For fixed $\Delta > 0$, real t , and arbitrary sufficiently small Hermitian $X \in M_m(\mathbb{C})$,*

$$E_m(e^{iX}) - E_m(I) = \frac{\hat{\kappa}_m}{2} \text{Tr } X^2 + O(\|X\|^4), \quad \hat{\kappa}_m = \frac{2t^4(z_0^2 - 2t^2)}{mz_0(z_0^4 - 2z_0^2t^2 + 8t^4)}. \quad (8)$$

The remainder is local at identity, with constants allowed to depend on the fixed model parameters. No independent $(\text{Tr } X)^2$ term occurs. The stiffness is positive for $t \neq 0$, vanishes for $t = 0$, and has energy units. Writing $\varepsilon = t/\Delta$,

$$\hat{\kappa}_m = \frac{\Delta}{m} [2\varepsilon^4 - 12\varepsilon^6 + O(\varepsilon^8)] = \frac{2t^4}{m\Delta^3} + O(t^6/\Delta^5). \quad (9)$$

Its full parameter dependence is homogeneous of degree one in (t, Δ) and is specified by (7); in particular $\hat{\kappa}_m \sim |t|/(8m)$ as $|t|/\Delta \rightarrow \infty$ with Δ fixed.

Proof. Let the eigenvalues of X be x_a . Put $N(z) = z(z^2 - 2t^2)$ and $D_0(z) = z^2(z^2 - 4t^2)$. The scalar root is simple, so the implicit-function theorem applies near identity. The exact secular formula expands as

$$R_{e^{iX}}(z) = R_I(z) - \frac{t^4 N(z)}{m D_0(z)^2} \sum_a x_a^2 + O(\|X\|^4), \quad R_I(z) = \frac{z^2 - 2t^2}{z(z^2 - 4t^2)}.$$

Moreover

$$R'_I(z) = -\frac{z^4 - 2z^2t^2 + 8t^4}{z^2(z^2 - 4t^2)^2} < 0 \quad (z > 2|t|).$$

Substitution of $z = z_0 + \delta z$ into $\Delta R = 1$ gives $\delta z = t^4 N(z_0) \text{Tr } X^2 / [m D_0(z_0)^2 R'_I(z_0)] + O(\|X\|^4)$. Since $E = \Delta - z$, simplification proves (8). Conjugation invariance alone would allow both $\text{Tr } X^2$ and $(\text{Tr } X)^2$. The additional property here is the normalized sum of independent one-ring resolvents: the exact secular equation is unchanged by reversing any individual eigenangle x_a . All mixed quadratic monomials $x_a x_b$ with $a \neq b$ are therefore absent. This excludes a trace-square contribution for $m \geq 2$; for $m = 1$ the two proposed invariants coincide. This argument does not exclude multi-trace terms at fourth and higher orders in X .

Because $z_0 > 2|t|$, both the numerator and denominator in the stated stiffness are positive for nonzero t . Power counting gives energy units and the scaling under $(t, \Delta) \mapsto (ct, c\Delta)$. Expansion of the physical cubic root gives $z_0/\Delta = 1 + 2\varepsilon^2 - 8\varepsilon^6 + O(\varepsilon^8)$, which yields the weak-hopping series. For large hopping, the identity top-mode weight is $1/4$, so $z_0 = 2|t| + \Delta/4 + O(\Delta^2/|t|)$. Substitution gives the final asymptotic statement. $\square$

Corollary 2.4 (All-orders static two-block Hessian constraint). *For independent frozen-link blocks define $V(W_3, W_2) = E_3(W_3) + E_2(W_2) - E_3(I) - E_2(I)$. Write small Hermitian logarithms in the manuscript's integral central normalization as $X_3 = X_3^0 - 2fI_3$ and $X_2 = X_2^0 + 3fI_2$, where $\text{Tr } X_m^0 = 0$. Then its exact quadratic Taylor term is*

$$V = \frac{1}{2} [\hat{\kappa}_3 \text{Tr}[(X_3^0)^2] + \hat{\kappa}_2 \text{Tr}[(X_2^0)^2] + \hat{\kappa}_1 f^2] + O(\|(X_3, X_2)\|^4), \quad \hat{\kappa}_1 = 12\hat{\kappa}_3 + 18\hat{\kappa}_2.$$

If $t_2 = t_3 \neq 0$ and $\Delta_2 = \Delta_3$, then $\hat{\kappa}_2/\hat{\kappa}_3 = 3/2$ exactly. Unequal parameters obey the exact formula (8) separately in each block; their ratio is not generally the fourth-order power law away from weak hopping.

Proof. Tracelessness removes cross terms and gives $\text{Tr } X_3^2 = \text{Tr}[(X_3^0)^2] + 12f^2$ and $\text{Tr } X_2^2 = \text{Tr}[(X_2^0)^2] + 18f^2$. Add the two exact Hessians. For equal parameters, z_0 is identical in both blocks and their stiffnesses differ only by $1/m$. $\square$

“All orders” here means that the exact small-curvature Hessian includes every hopping order in this specified static elimination. It does not mean that the full potential is exactly a Wilson trace action, nor that electric dressing, extra auxiliary representations, new interactions, or quantum renormalization preserve the relation.

2.2.3 Fourth-order interaction and an explicit uniform remainder

Theorem 2.5 (Generated non-Abelian loop coefficient with a remainder). *For $|t_m|/\Delta_m = |\varepsilon_m| < 1/4$, uniformly in $W_m \in U(m)$,*

$$E_m(W_m) = -\frac{2t_m^2}{\Delta_m} + \frac{2t_m^4}{m\Delta_m^3} [m - \text{Re Tr } W_m] + R_m(W_m), \quad |R_m(W_m)| \leq \frac{2048\Delta_m|\varepsilon_m|^6}{1 - 16|\varepsilon_m|^2}. \quad (10)$$

Thus independent blocks have leading magnetic coefficients

$$\kappa_m^{\text{loop}} = \frac{2t_m^4}{m\Delta_m^3}, \quad \kappa_1^{\text{loop}} = 12\kappa_3^{\text{loop}} + 18\kappa_2^{\text{loop}}. \quad (11)$$

The explicit remainder constant is conservative; its role is to give a proved uniform estimate, not an optimized numerical error prediction.

Proof. Write $P = |s\rangle\langle s|$, $Q = I - P$, and $V = -K$. The scalar Feshbach equation is

$$E = \langle s, VQ(E - \Delta - QVQ)^{-1}QVs \rangle.$$

Put $a_2 = \langle s, VQVs \rangle$ and $a_4 = \langle s, VQVQVQVs \rangle$. The expansion through fourth order, including the energy dependence of the resolvent, gives

$$E = -a_2/\Delta - a_4/\Delta^3 + a_2^2/\Delta^3 + O(t^6/\Delta^5).$$

The two unitary backtracks give $a_2 = 2t^2$. A two-step root return is in the singlet and is removed by the intermediate Q . The remaining four-step paths are two retracing excursions and the two oriented full cycles. Therefore

$$a_4 = t^4 \left(2 + \frac{2}{m} \operatorname{Re} \operatorname{Tr} W \right).$$

The folded term a_2^2/Δ^3 must be included: it cancels the holonomy-independent fourth-order contribution at $W = I$ and gives exactly the coefficient displayed above.

For an explicit error bound, hold the unitary links fixed and allow the hopping parameter ε to be complex. The unperturbed matrix has spectrum $\{0, \Delta\}$ and the perturbation norm is at most $2\Delta|\varepsilon|$. For $|\varepsilon| < 1/4$ the two spectral disks are disjoint. A Riesz contour gives a single analytic eigenvalue continuing zero, and spectral inclusion for a perturbation of a normal matrix bounds it by $|E(\varepsilon)| \leq 2\Delta|\varepsilon|$. Position parity conjugates the Hamiltonian at ε to that at $-\varepsilon$ and fixes P , so this eigenvalue is even. On a circle $|\varepsilon| = r < 1/4$, Cauchy's estimate bounds its n th Taylor coefficient by $2\Delta r^{1-n}$. Taking $r \uparrow 1/4$ gives $|c_n| \leq (\Delta/2)4^n$. Summing the even tail from $n = 6$ gives the stated bound, uniformly in W and in m . $\square$

At the next hopping order, direct expansion of the same exact secular formula gives, with $c_W = m^{-1} \operatorname{Re} \operatorname{Tr} W$,

$$E_m(W)/\Delta = -2\varepsilon^2 + 2(1 - c_W)\varepsilon^4 + (12c_W - 4)\varepsilon^6 + O(\varepsilon^8).$$

In particular $E_m(I) = -2\Delta\varepsilon^2 + 8\Delta\varepsilon^6 + O(\Delta\varepsilon^8)$: subtracting only the second-order constant is not an exact normalization of the full potential. These higher static orders change $\hat{\kappa}_m$ but do not generate an independent central quadratic invariant.

For unequal positive edge hoppings $t_{j,m}$ and positive penalties $\Delta_{1,m}, \Delta_{2,m}, \Delta_{3,m}$ at the nonroot positions, the leading oriented-loop coefficient remains

$$\kappa_m^{\text{loop}} = \frac{2 \prod_{j=0}^3 t_{j,m}}{m \Delta_{1,m} \Delta_{2,m} \Delta_{3,m}}. \quad (12)$$

The two shortest oriented cycles have those three intermediate positions; all fourth-order backtracking and folded terms are holonomy independent. A positive root-nonsinglet penalty is still needed for an isolated unperturbed band, although those shortest cycles do not visit it. This last leading-order statement does not extend the equal-penalty exact secular formula or its explicit remainder constant to a different Hamiltonian. None of these coefficients is the compact-rotor coefficient under a parameter renaming: the models have different retained states and virtual paths.

2.3 A controlled finite-volume effective Hamiltonian

The following theorem replaces a formal electric-energy power count by a global operator-norm estimate for a precisely specified regularization. It leaves link multiplication uncompressed. Its price is a bounded electric dispersion at large representation labels; the unbounded Laplacian is treated separately below.

Let $(\mathcal{V}, \mathcal{E}, \mathcal{P})$ be a finite oriented cell complex whose designated faces are simple four-edge cycles, each with a chosen root. Faces may share edges. Set $G = G_{\text{src}}$ and

$$\mathcal{H} = L^2(G^{\mathcal{E}}, dg) \otimes \bigotimes_{a \in \mathcal{I}} \mathcal{K}_{m_a}, \quad \mathcal{I} = \{(p, m) : p \in \mathcal{P}, m = 3, 2\}.$$

More generally a finite list of defining-carrier auxiliaries may be used. Each $\mathcal{K}_{m_a}$ is the $4m_a^2$ -dimensional space in (1); different auxiliaries have independent tensor factors. Link matrices are their exact unitary multiplication operators. With $P_a = |s_a\rangle\langle s_a|$, put

$$P = I_{\text{link}} \otimes \bigotimes_a P_a, \quad Q = I - P, \quad A = \sum_a d_a (I - P_a), \quad d_a \geq 1,$$

where identities on other factors are understood. Thus $AP = 0$ and $A \geq Q$. The retained space $P\mathcal{H}$ is canonically $\mathcal{H}_{\text{link}}$, with all auxiliaries in their root singlets. The positive parameter Δ has energy units; the individual gaps and hoppings are $\Delta_a = \Delta d_a$ and $t_a = \Delta \varepsilon \tau_a$, respectively, with real dimensionless τ_a . Define

$$B = - \sum_a \tau_a \sum_{j=0}^3 (T_{j,a} + T_{j,a}^*), \quad b = 2 \sum_a |\tau_a|, \quad \|B\| \leq b.$$

The dimensionless electric operator on edge e is

$$D_e = \frac{1}{2} (\beta_3 \mathbf{C}_{3,e} + \beta_2 \mathbf{C}_{2,e} + \beta_1 \mathbf{N}_e^2), \quad \beta_i > 0.$$

For explicit positive caps K_e , functional calculus gives

$$D_K = \sum_e \min(D_e, K_e), \quad d = \|D_K\| \leq \sum_e K_e, \quad H_K(\varepsilon) = \Delta (A + \varepsilon B + \varepsilon^4 D_K). \quad (13)$$

D_K acts trivially on the auxiliaries. All terms are bounded and self-adjoint, so the domain is the entire Hilbert space. The cap preserves left and right gauge transformations and every allowed quotient-group representation. No fundamental link is compressed or declared unitary after truncation. H_K is gauge invariant under simultaneous link and auxiliary transformations; invariant sewing can restrict the theorem to its Gauss sector. The caps are additional regulator parameters, not derived physical constants.

The same theorem and proof also apply to a bounded nonnegative self-adjoint gauge-invariant electric operator D satisfying $[D, P] = [D, \Gamma] = 0$, with D_K in the target replaced by PDP and $d = \|D\|$. Here Γ is the total auxiliary position parity used in the proof. This variant allows gauge-reduced electric energies that depend on auxiliary position, provided they preserve the retained sector and parity.

Theorem 2.6 (Explicit sixth-order norm remainder and dressed dynamics). *Choose $r > 0$ with $br + dr^4 \leq 1/8$ and $0 \leq \varepsilon < r$. One explicit choice is*

$$r = \min\{1, (16b)^{-1}, (16d)^{-1/4}\},$$

omitting a term with zero denominator. There is a gauge-equivariant unitary $\mathcal{U}_\varepsilon$, with $\mathcal{U}_0 = I$, mapping $P\mathcal{H}$ onto the isolated spectral cluster of $H_K(\varepsilon)$ continuing its unperturbed zero sector. Its separation from the complementary cluster is at least

$$\Delta [1 - 2(b\varepsilon + d\varepsilon^4)] \geq 3\Delta/4.$$

In the canonical direct-rotation convention, set

$$H_{\text{eff}} = P\mathcal{U}_\varepsilon^* H_K(\varepsilon) \mathcal{U}_\varepsilon P.$$

On $P\mathcal{H} \simeq \mathcal{H}_{\text{link}}$ define the full target

$$\begin{aligned} H_{\text{target}} &= \Delta\varepsilon^4 D_K - 2\Delta\varepsilon^2 \sum_a \frac{\tau_a^2}{d_a} I \\ &\quad + \Delta\varepsilon^4 \sum_a \frac{2\tau_a^4}{m_a d_a^3} [m_a - \text{Re Tr } W_a]. \end{aligned} \quad (14)$$

Then

$$\|H_{\text{eff}} - H_{\text{target}}\| \leq \mathcal{R}_6 := \frac{4\Delta}{3} \frac{(\varepsilon/r)^6}{1 - (\varepsilon/r)^2}. \quad (15)$$

In particular, for $\varepsilon \leq r/2$ this is $\|H_{\text{eff}} - H_{\text{target}}\| \leq (16/9)\Delta r^{-6}\varepsilon^6$. For every real physical time T ,

$$\left\| e^{-iH_K T/\hbar} \mathcal{U}_\varepsilon P - \mathcal{U}_\varepsilon e^{-iH_{\text{target}} T/\hbar} P \right\| \leq \min\{2, |T|\mathcal{R}_6/\hbar\}. \quad (16)$$

Thus an error tolerance $\eta < 2$ is guaranteed for $|T| \leq \eta\hbar/\mathcal{R}_6$ for states prepared in the dressed cluster.

Proof. We provide the norm estimate, coefficient calculation, and preparation convention explicitly. Consider the bounded analytic family $F(z) = A + zB + z^4 D_K$ for complex z . On the circle $|\zeta| = 1/2$, $\|(\zeta - A)^{-1}\| \leq 2$ since $\text{Spec } A \subset \{0\} \cup [1, \infty)$. For $|z| \leq r$, put $v = b|z| + d|z|^4 \leq 1/8$. The resolvent Neumann series gives

$$\|(\zeta - F(z))^{-1}\| \leq \frac{2}{1 - 2v} \leq \frac{8}{3}.$$

Its Riesz projection $P(z)$ on this circle obeys, by the resolvent identity and the circle length,

$$\delta(z) := \|P(z) - P\| \leq \frac{8}{3}v \leq \frac{1}{3}, \quad \|F(z)P(z)\| \leq \frac{1}{2\pi} \int_{|\zeta|=1/2} |\zeta| \|(\zeta - F(z))^{-1}\| |d\zeta| \leq \frac{2}{3}.$$

These estimates are independent of the dimension of the link Hilbert space and do not require a finite-rank retained projector. Define the analytic direct rotation

$$\mathcal{U}(z) = [P(z)P + (I - P(z))(I - P)][I - (P(z) - P)^2]^{-1/2}, \quad (17)$$

using the power-series square root at I . Direct multiplication shows $P(z)\mathcal{U}(z) = \mathcal{U}(z)P$ and

$$\|\mathcal{U}(z)\|, \|\mathcal{U}(z)^{-1}\| \leq \frac{1 + \delta(z)}{\sqrt{1 - \delta(z)^2}} \leq \sqrt{2}.$$

For real z , $P(z)$ is orthogonal and this rotation is unitary. Consequently

$$F_{\text{eff}}(z) := P\mathcal{U}(z)^{-1}F(z)\mathcal{U}(z)P \quad \text{satisfies} \quad \|F_{\text{eff}}(z)\| \leq 4/3.$$

The product of all auxiliary position-parity operators, denoted Γ , commutes with A, D_K and equals I on $P\mathcal{H}$, whereas $\Gamma B \Gamma = -B$. Hence $F(-z) = \Gamma F(z) \Gamma$, and the contour and direct-rotation formulas imply $F_{\text{eff}}(-z) = F_{\text{eff}}(z)$. It has only even Taylor coefficients. Cauchy's estimate bounds its n th coefficient by $(4/3)r^{-n}$. Summing the coefficients of orders 6, 8, ... proves (15) once its second- and fourth-order coefficients are identified.

First turn off D_K . The family is then multiplication by a finite matrix for each configuration g . Its independent auxiliary summands act on different tensor factors, even on shared edges. Its lowest auxiliary eigenvector is the tensor product of the individual lowest vectors, and its eigenvalue is exactly their sum. The direct rotation identifies this rank-one fiber with the bare

product of singlets, so its effective Hamiltonian is precisely multiplication by that sum. The second- and fourth-order calculation of Theorem 2.5, including the folded term, gives

$$-2 \sum_a \frac{\tau_a^2}{d_a} z^2 + \sum_a \frac{2\tau_a^4}{m_a d_a^3} [m_a - \text{Re Tr } W_a] z^4.$$

Thus all mixed-auxiliary static terms cancel, rather than being omitted by power counting. The second-order constant has been retained; the fourth-order identity contribution cancels at $W_a = I$ because the folded term is included. Returning $z^4 D_K$ adds only $PD_K P = D_K$ at fourth order: it preserves P and cannot enter a coefficient with fewer than four powers of z . Every correction involving it and a hop is even in z , and the first possible such term has two hops and order six. This establishes the displayed target without dropping electric dressing or geometric terms at the claimed order; all of those terms are included in the bounded remainder.

For real ε , the spectral inclusion for a self-adjoint bounded perturbation places the low cluster in $[-v, v]$ and the remainder in $[1 - v, \infty)$, giving the stated gap. All contour constructions commute with the gauge representation, hence so does $\mathcal{U}_\varepsilon$. The exact intertwining of the dressed cluster reduces its dynamics to H_{eff} . Duhamel's formula for the two bounded self-adjoint effective Hamiltonians gives $|T| \|H_{\text{eff}} - H_{\text{target}}\|/\hbar$; the difference of two unitaries is also bounded by two. This proves (16). $\square$

At a magnetic evolution time $|T| = c\hbar/(\Delta\varepsilon^4)$, with fixed c, r , the right side of (16) is at most

$$\frac{4c}{3r^6} \frac{\varepsilon^2}{1 - (\varepsilon/r)^2}.$$

Hence the theorem controls a full order-one magnetic evolution with an error tending to zero quadratically in the hopping ratio, for fixed finite volume and fixed electric caps.

The estimate is deliberately conservative. For $N = |\mathcal{I}|$ auxiliaries with $|\tau_a| \leq \tau_*$, $b \leq 2N\tau_*$; for M links capped at K , $d \leq MK$. Thus its sufficient smallness condition and its constant worsen with volume and electric cap. The auxiliary representation ranks enter the coefficient and total dimension $\prod_a 4m_a^2$, but not the unitary-hopping norm bound. The numbers d_a must have a common positive lower bound; choosing $\Delta = \min_a \Delta_a$ makes the normalization explicit. No thermodynamic-uniform or continuum-uniform estimate is claimed. The theorem controls dressed preparations. For a bare singlet preparation the triangle inequality adds $2\|(\mathcal{U}_\varepsilon - I)P\|$ to (16); from the proof this preparation term is $O(b\varepsilon + d\varepsilon^4)$, rather than sixth order.

Remark 2.7 (Asymptotic control and numerical usefulness). The explicit bound is conservative. In the one-square compressed examples of Appendix A.3, $b = 2$ and the displayed choice gives $r = 1/32$. At $\varepsilon/r = 0.4$ one has $R_6/\Delta \simeq 0.00650159$; at one magnetic time $T = \hbar/(\Delta\varepsilon^4)$ the uncapped estimate $|T|R_6/\hbar \simeq 2.66305 \times 10^5$ makes (16) only the general upper bound of two. This does not contradict its vanishing error as $\varepsilon \rightarrow 0$ at fixed volume, cap, and r , nor the much smaller computed errors. It distinguishes a rigorous asymptotic estimate from a useful accuracy guarantee at the plotted parameters.

2.3.1 A finite Peter–Weyl version with its boundary terms retained

For finite-matrix calculations choose on every edge a finite collection Λ_e of full Peter–Weyl summands $V_\lambda \otimes V_\lambda^*$ allowed by G , and let Π_Λ project onto their product, acting trivially on auxiliaries. It preserves both endpoint gauge actions. Put $B_\Lambda = \Pi_\Lambda B \Pi_\Lambda$ and either compress D_K or use the ordinary electric operator restricted to this finite space. In the latter case its exact finite norm is d_Λ ; use it for d in the theorem. All operators now act on a finite-dimensional space. The norm and parity proof remains unchanged, but the target coefficient is generally different near its representation boundary.

Here is an explicit target requiring no unitarity assumption for compressed links. On the compressed space retain the notation P, Q, A , and set $R = Q(A|_{Q\mathcal{H}})^{-1}Q$. Define

$$S_\Lambda = PB_\Lambda RB_\Lambda P, \quad K_\Lambda = PB_\Lambda R^2 B_\Lambda P, \quad U_\Lambda = PB_\Lambda RB_\Lambda RB_\Lambda RB_\Lambda P. \quad (18)$$

The rigorously controlled finite-matrix target is

$$H_{\text{target},\Lambda} = \Delta \left[-\varepsilon^2 S_\Lambda + \varepsilon^4 (D_\Lambda - U_\Lambda + \tfrac{1}{2}\{K_\Lambda, S_\Lambda\}) \right]. \quad (19)$$

It satisfies the same bound (15). This formula includes constant shifts, folded terms, and every four-hop process, including any mixed-face process induced by the compression.

Derivation of the finite-matrix coefficient. $PB_\Lambda P = 0$. Write the low subspace as the graph of $T(z) : P\mathcal{H} \rightarrow Q\mathcal{H}$ and solve its invariance equation order by order. With D_Λ temporarily zero,

$$\begin{aligned} T_1 &= -RB_\Lambda P, \\ T_2 &= RB_\Lambda RB_\Lambda P, \\ T_3 &= -RB_\Lambda RB_\Lambda RB_\Lambda P + R^2 B_\Lambda P S_\Lambda. \end{aligned}$$

The unnormalized effective operator $zPB_\Lambda T(z)$ has second coefficient $-S_\Lambda$ and fourth coefficient $-U_\Lambda + K_\Lambda S_\Lambda$. Normalizing the graph by $[P + T^*T]^{-1/2}$ conjugates it by $[P + T^*T]^{1/2}$ and adds $\frac{1}{2}[K_\Lambda, -S_\Lambda]$ at fourth order. The result is $-U_\Lambda + \frac{1}{2}\{K_\Lambda, S_\Lambda\}$, as claimed. The graph normalization is the restriction of the direct rotation. The electric term contributes D_Λ at fourth order as before, and parity removes odd coefficients. The resolvent proof bounds the remaining even coefficients. $\square$

For example, the exact second-order boundary correction relative to compressing the uncut answer is the nonnegative operator

$$\mathcal{B}_{2,\Lambda} = P\Pi_\Lambda B(I - \Pi_\Lambda)RB\Pi_\Lambda P, \quad 0 \leq \mathcal{B}_{2,\Lambda} \leq b^2 P. \quad (20)$$

At fourth order the exact boundary correction is $-U_\Lambda + \{K_\Lambda, S_\Lambda\}/2$ minus the compression of its uncut counterpart; the displayed finite polynomials calculate it, and its norm is at most $4b^4$ since $\|R\| \leq 1$. Boundary effects can therefore begin at second order, parametrically before the magnetic term. It is incorrect to replace (19) globally by a Wilson expression while treating compressed fundamental links as unitary.

There is also an exact, limited padding statement. For a finite input representation window Λ_0 , close its edge labels under all fundamental and dual tensor products reachable by at most four hopping steps. Each such product has a finite decomposition into irreducibles, so the resulting finite collection Λ exists. Every intermediate sector of every coefficient word in (18) then belongs to Λ . Since R acts only on the auxiliaries, inserting Π_Λ between factors does not change those words on the input window. In particular their Π_{Λ_0} matrix blocks equal the uncut Wilson coefficients through fourth order. This is equality of specified coefficient matrix elements, not a global cutoff-uniform norm approximation or a long-time confinement statement for that input window. Computing the full finite polynomial and the exact finite low spectral cluster tests the stronger global remainder statement without requiring any such confinement claim.

2.3.2 Unbounded electric motion and the missing uniform estimate

The original unsaturated Hamiltonian is

$$H_\infty = H_E + \sum_a H_a(g), \quad H_E = \tfrac{1}{2} \sum_e [\alpha_3 \mathbf{C}_{3,e} + \alpha_2 \mathbf{C}_{2,e} + \alpha_1 \mathbf{N}_e^2], \quad \alpha_i > 0. \quad (21)$$

It is self-adjoint on the Sobolev domain of the elliptic electric Laplacian, bounded below, and has compact resolvent and finite heat trace in finite volume: the remaining finite auxiliary matrices and hopping are bounded perturbations on a compact configuration manifold. These facts do not give an operator-norm auxiliary-elimination expansion. The electric spectrum is unbounded, so high electric states in the retained sector overlap auxiliary-excited energies. There is no single global cluster around zero separated from all of them by the auxiliary gap. In contrast, the saturated full-space model has no finite-heat-trace claim: a bounded Hamiltonian on its infinite-dimensional space cannot have a trace-class positive heat operator.

To compare the two models quantitatively, set $\alpha_i = \Delta \varepsilon^4 \beta_i$ and $D_\infty = \sum_e D_e$. The capped and uncapped electric operators agree on a bare representation window with $D_e \leq K_e$ for every edge. That window need not be preserved by hopping. For any initial ψ in the domain of H_∞ , Duhamel's formula gives the explicit tail requirement

$$\|(e^{-iH_\infty T/\hbar} - e^{-iH_K T/\hbar})\psi\| \leq \frac{\Delta \varepsilon^4}{\hbar} \int_0^{|T|} \|(D_\infty - D_K)e^{-iH_\infty s/\hbar}\psi\| ds, \quad (22)$$

with the sign of s reversed for negative T . An energy-window theorem for the unbounded model needs an estimate of this weighted high-representation tail, together with compatible auxiliary dressing estimates, uniform in the claimed window, cutoff, volume, and time range. No such bound follows from the finite auxiliary gap alone. Removing the cap, taking thermodynamic limits, and taking a continuum limit are therefore separate open problems here.

Finally the magnetic coefficients in (14) and the exact static Hessian do not fix $\beta_3, \beta_2, \beta_1$. Even in a weak-field limit, wave speeds involve products of electric and magnetic coefficients, whereas coupling normalizations involve their ratios. Equal limiting speeds and a Lorentz-invariant continuum require additional dynamical matching. An exact static stiffness identity supplies neither by itself.

2.4 Three coefficient conventions

The same central weights occur in three different constructions. Their coefficients must not be identified without a matching calculation:

Notation	Definition	Status
κ_m	Positive rejection-cost normalization in (56).	Chosen primitive protocol coefficients; not derived singlet energies.
κ_m^{loop}	$2t_m^4/(m\Delta_m^3)$ in (11).	Derived fourth-order weak-hopping coefficient, not the exact stiffness at finite hopping.
$\hat{\kappa}_m$	Exact identity Hessian in (8).	Derived frozen-link coefficient including every hopping order of this Hamiltonian.

The primitive cost has its own relation $\kappa_1 = 12\kappa_3 + 18\kappa_2$ because that cost is chosen as two single traces. The separately proved exact relation $\hat{\kappa}_1 = 12\hat{\kappa}_3 + 18\hat{\kappa}_2$ concerns the frozen-link singlet Hamiltonian and does not turn primitive costs into derived energies. Its content is the exclusion of an additional trace-square term by exact elimination. Gauge symmetry alone allows such a term. The electric coefficients $\beta_3, \beta_2, \beta_1$ remain independent inputs; none of these relations predicts observed gauge couplings.

2.5 Quantitative consequences and correction classes

The two-block frozen-link model consequence is exact at the Hessian level. Define $F(t, \Delta) = m\hat{\kappa}_m(t, \Delta)$ using (8); F is independent of m . For nonzero hoppings,

$$\frac{\hat{\kappa}_2}{\hat{\kappa}_3} = \frac{3}{2} \frac{F(t_2, \Delta_2)}{F(t_3, \Delta_3)}, \quad \frac{\kappa_2^{\text{loop}}}{\kappa_3^{\text{loop}}} = \frac{3}{2} \left(\frac{t_2}{t_3} \right)^4 \left(\frac{\Delta_3}{\Delta_2} \right)^3. \quad (23)$$

The second equality concerns the leading coefficients only. Its relative error as a formula for the exact ratio is $O((t_2/\Delta_2)^2 + (t_3/\Delta_3)^2)$ as both ratios tend to zero. Equal gaps and equal nonzero hopping give 3/2 exactly, without a small-hopping condition, but their equality is a microscopic assumption. The normalized stiffness obeys $m\hat{\kappa}_m/(2\Delta_m\varepsilon_m^4) = 1 - 6\varepsilon_m^2 + O(\varepsilon_m^4)$.

Change or diagnostic	Proved consequence	Scope
Higher hopping orders of the same independent frozen-link model	Change $\hat{\kappa}_3, \hat{\kappa}_2$ but preserve $\hat{\kappa}_1 - 12\hat{\kappa}_3 - 18\hat{\kappa}_2 = 0$	Exact Hessian; full finite-curvature potential need not be Wilson.
Unequal uniform block parameters	Exact secular/Hessian formulas remain valid separately	Equal-parameter ratio is replaced by the first formula in (23).
Electric motion at order ε^4	All dressing first enters the remainder beginning at order ε^6	Norm bound of Theorem 2.6; no protected quantum stiffness identity.
Hard representation compression	Boundary terms can appear already at order ε^2	Use the full polynomial (19), or the proved padded coefficient window.
Other auxiliary representations or new interactions	May supply an independent central invariant	Gauge covariance alone does not impose the two-block relation.
Measured nonzero static residual	Inconsistent with the exact isolated independent singlet model after calibration	Does not reject all gauge theories or SMM/DTT.

No numerical check is used to establish ground-state uniqueness, isolation, or an analytic remainder: those statements have proofs above. The reproducible calculations compare the secular equation, direct matrix spectra, central and traceless finite-difference Hessians, and the exact effective block with their independent analytic expressions. The finite-matrix dynamics example is a compact $U(1)$ implementation of the reduction theorem, not a numerical construction of the full $S(U(3) \times U(2))$ quantum field theory.

Even in a weak-field continuum approximation to $H = \frac{1}{2} \int (aE^2 + bB^2)$, the Hamilton equations give a squared propagation speed proportional to ab , while temporal rescaling makes the coupling normalization proportional to $\sqrt{a/b}$ (with lattice spacing and $\hbar$ restored according to convention). Each gauge sector has its own electric coefficient. A relation among magnetic Hessians therefore does not select physical gauge couplings, equal propagation speeds, or Lorentz symmetry. Identifying these model parameters with measured physics is a separate matching problem.

3 Companion rotor model: generated cycles and circulation obstruction

We now specify a Hamiltonian with no bare plaquette term. Its intermediate states can carry incomplete endpoint charges; completion returns to the neutral sector. Virtual closed transfers then produce magnetic loop energies. This is a concrete realization of a proposed temporal interface, not a derivation of the Hamiltonian from the finite clock. Rotor and virtual-loop mechanisms are established in emergent and simulated gauge theory [16, 30, 31].

3.1 Link and vertex registers

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite oriented simple graph. Choose incidence matrix D with $+1$ at the tail and -1 at the head of each edge. The Hilbert space is

$$\mathcal{H} = L^2(U(1)^{\mathcal{E}} \times U(1)^{\mathcal{V}}), \quad |n, m\rangle, \quad n \in \mathbb{Z}^{\mathcal{E}}, \quad m \in \mathbb{Z}^{\mathcal{V}}.$$

Link multiplication $U_e = e^{i\theta_e}$ raises n_e by one. A vertex raising operator $b_v^\dagger = e^{i\phi_v}$ raises m_v by one. For $e = (u \rightarrow v)$ define

$$T_e = U_e b_u^\dagger b_v, \quad H = \frac{\Delta}{2} \sum_v m_v^2 + \frac{\kappa}{2} \sum_e n_e^2 - \frac{\lambda}{2} \sum_e (T_e + T_e^*), \quad (24)$$

where $\Delta > 0$, $\kappa \geq 0$, and $\lambda \geq 0$ have energy units. The hopping is bounded, so this Hamiltonian is self-adjoint on the domain of its nonnegative diagonal part and is bounded below.

Proposition 3.1 (Exact constraint preservation and reduced register). *The commuting operators $\mathcal{G}_v = m_v - (Dn)_v$ commute with H . In their common zero sector, the isometry $I|n\rangle = |n, m = Dn\rangle$ gives the exact reduced Hamiltonian*

$$H_{\text{red}} = \frac{\Delta}{2} \|Dn\|^2 + \frac{\kappa}{2} \|n\|^2 - \frac{\lambda}{2} \sum_e (S_e + S_e^*), \quad S_e |n\rangle = |n + \mathbf{e}_e\rangle. \quad (25)$$

Proof. T_e changes n by $\mathbf{e}_e$ and m by $D\mathbf{e}_e$; it therefore leaves every $\mathcal{G}_v$ unchanged. The diagonal terms also commute with them. Every integer link assignment has exactly one vertex assignment in the zero sector, namely $m = Dn$. Substitution and the transfer rule give the reduced Hamiltonian. $\square$

The Hamiltonian preserves all charge sectors. Selection of the zero sector uses invariant sewing or a specified initial condition; it does not follow from commutation alone. At small λ/Δ , the subspace $Dn = 0$ is the unperturbed completed sector. This is distinct from the exact Gauss sector, which also contains states with $m = Dn \neq 0$.

3.2 The first contribution of a simple cycle

Set $\kappa = 0$ for the following coefficient calculation. A proper nonempty subset S of the edges of an oriented simple cycle has $c(S)$ occupied path components. Its charge configuration has two unit endpoints per component and energy $\Delta c(S)$. Let W_C shift n by one oriented circulation around a cycle C of length L .

Theorem 3.2 (Generated leading-cycle coefficient). *At its first possible perturbative order, the contribution proportional to W_C and its reverse is*

$$H_C^{(L)} = - \binom{2L-2}{L-1} \frac{(\lambda/2)^L}{\Delta^{L-1}} (W_C + W_C^*). \quad (26)$$

This is a finite-order coefficient at $\kappa = 0$, not a claim of a convergent thermodynamic expansion or of a Coulomb phase.

Proof. Reaching the target circulation in L transfers requires each of its L edges once in the chosen orientation. Every proper intermediate subset is charged, so no intermediate return to the completed sector occurs and folded neutral-return terms cannot produce this circulation at this order. For a proper subset set

$$F(S) = \sum_{\text{orders of } S} \prod_{k=1}^{|S|} \frac{1}{c(S_k)}, \quad F(\emptyset) = 1.$$

Removing the last transfer gives $F(S) = c(S)^{-1} \sum_{e \in S} F(S \setminus \{e\})$. If the occupied components have lengths $n_1, \dots, n_c$, induction and the Catalan recurrence $C_n = \sum_{r=0}^{n-1} C_r C_{n-1-r}$, $C_0 = 1$, give $F(S) = \prod_j C_{n_j}$. Indeed deleting an edge in component j splits it into two paths, and summing its positions gives C_{n_j} ; summing over j cancels the factor $c(S)$. Choosing the last edge of the full cycle leaves one path of length $L - 1$. Thus the denominator sum is $LC_{L-1} = \binom{2L-2}{L-1}$. There are L negative hoppings and $L - 1$ negative energy resolvents, giving total sign minus and the energy power displayed. Reverse orientation gives the adjoint term. $\square$

For a square, sixteen transfer orders have three intermediate costs Δ , and eight have costs $\Delta, 2\Delta, \Delta$. Their dimensionless denominator sum is $16 + 8/2 = 20$. Hence

$$H_{\square}^{(4)} = -\frac{5\lambda^4}{4\Delta^3}(W_{\square} + W_{\square}^*), \quad (27)$$

$$K_4 = \frac{5\lambda^4}{2\Delta^3}, \quad K_6 = \frac{63\lambda^6}{8\Delta^5}, \quad \frac{K_6}{K_4} = \frac{63}{20} \left(\frac{\lambda}{\Delta}\right)^2. \quad (28)$$

Here K_L is the coefficient of $1 - \cos \theta_C$, after adding an irrelevant constant. The square coefficient agrees with an existing compact-rotor implementation after matching the charging-energy convention [33]; it is not claimed as a newly discovered ring exchange. The ratio concerns the *first coefficient of a specified six-edge circulation*. It is not the whole sixth-order magnetic response: square renormalizations, other paths, and finite- κ denominators must be computed separately.

3.3 Normalizable finite-region records require flux control

Theorem 3.3 (A circulation symmetry obstructs an induced electric penalty). *If $D\ell = 0$ for a nonzero integer circulation, define $W_{\ell}|n\rangle = |n + \ell\rangle$. At $\kappa = 0$,*

$$[H_{\text{red}}, W_{\ell}] = 0, \quad [||n||^2, W_{\ell}][n] = (2n \cdot \ell + ||\ell||^2)|n + \ell\rangle.$$

Any exact reduction intertwining these translations preserves their symmetry. It therefore cannot generate a standalone positive $U||n||^2/2$ term while all remaining terms commute with the translations.

Proof. The diagonal imbalance energy is unchanged by $n \mapsto n + \ell$, and all the link shifts commute. The displayed commutator follows by evaluating the number-square operator in the integer basis. Compressions and complementary resolvents inherit a commuting symmetry whenever their projection and domains preserve it. $\square$

Corollary 3.4 (Finite thermal trace distinguishes zero and positive flux penalty). *On a finite graph with a nonzero integer circulation, $\text{Tr } e^{-\beta H_{\text{red}}} = \infty$ at $\kappa = 0$ for every $\beta > 0$. For $\kappa > 0$ the Hamiltonian has compact resolvent and*

$$\text{Tr } e^{-\beta H_{\text{red}}} \leq e^{\beta|\mathcal{E}|\lambda} \left(\sum_{j \in \mathbb{Z}} e^{-\beta\kappa j^2/2} \right)^{|\mathcal{E}|} < \infty. \quad (29)$$

Proof. At zero penalty all diagonal heat-kernel entries along $n + j\ell$, $j \in \mathbb{Z}$, are equal by the translation symmetry. They are strictly positive because $e^{-\beta H}$ is a positive injective operator. Summing this infinite orthonormal sequence gives an infinite trace. For positive κ the diagonal operator dominates $\kappa\|n\|^2/2$ and has compact resolvent. The hopping norm is at most $|\mathcal{E}|\lambda$. Bounded perturbation and the min–max principle bound its heat trace by the displayed product. $\square$

If normalizable equilibrium in finite completed regions is required, the unmodified $\kappa = 0$ cyclic rotor model is excluded. Within (24), every positive κ works: normalization does not select its value. Other circulation-confining energies can also work, so this is not a uniqueness theorem for the quadratic penalty. The condition concerns an unbounded flux register on a graph with cycles; it is not a theorem about every gauge realization. Hard flux cutoffs break the exact translations and cannot be used to infer a physical electric stiffness of the untruncated zero-penalty model.

3.4 Inner histories and a positive imaginary-time sum

For $\lambda \geq 0$, expand the semigroup of $H_{\text{red}} = V(n) - \frac{\lambda}{2} \sum_e (S_e + S_e^*)$ in its hopping. A matrix element is a sum over integer-register paths, with r jumps at ordered times $0 < s_1 < \dots < s_r < \beta$ and weight

$$\left(\frac{\lambda}{2}\right)^r \exp \left[- \int_0^\beta V(n(s)) \, ds \right].$$

The bounded hopping ensures the Dyson expansion converges on finite time intervals. Each path is a generative transfer history; fixing its completed endpoints produces the semigroup matrix element. This is an exact representation of the specified Hamiltonian. It does not select Δ, κ, λ , turn imaginary time into an extra physical clock, or demonstrate a four-dimensional mass gap.

3.5 What this model establishes

Incomplete charge configurations mediate a calculated magnetic term; finite-region normalization motivates control of otherwise unconfined circulations; and constraint preservation is exact. These are three different statements. The model leaves the graph, charge sector, Hamiltonian, and parameters as inputs. A photon phase, relativistic dispersion, and a continuum limit require separate estimates. In a three-dimensional weak-field Maxwell regime with renormalized electric and magnetic coefficients κ_R, K_B , the conventional lattice normalization gives $c_{\text{eff}} = a\sqrt{\kappa_R K_B}/\hbar$ and $g_{\text{eff}}^2 = \sqrt{\kappa_R/K_B}$. These are matching relations under that regime, not numerical predictions of the finite source. The freedom in their ratio is consistent with Proposition 5.4.

Part II

Conditional temporal and operational reconstruction

4 Conditional reconstruction and dependency ledger

The dependencies branch rather than form one chain:

$$\text{source class} + \text{minimal regular clock} \implies \mathbb{H}, N = 7, E_3, \quad (30)$$

$$\text{real marked cut} + \text{scalar extension} \implies E_2, \Delta_2 = 1, \quad (31)$$

$$\text{seven-clock chords} + \text{common response normalization} \implies r_7, \quad (32)$$

$$\text{record map} + \text{retained global tensor} \implies G_{\text{src}}, \quad (33)$$

$$\text{unframed records} + \text{faithful comparison} \implies C_{xy}, \quad (34)$$

$$\text{transport} + \text{R4 smooth reconstruction hypotheses} \implies \nabla, \quad (35)$$

$$\text{future-test sufficiency} + \text{coherent histories} \implies \text{retained transport}, \quad (36)$$

$$\text{invariant amplitude sewing} \implies \text{Gauss sector}, \quad (37)$$

$$\text{specified virtual-transfer Hamiltonian} \implies \text{calculated loop energy}, \quad (38)$$

$$\text{displacement/duality hypotheses} \implies \text{conditional dimension tests}, \quad (39)$$

$$3 + 2 \text{ carrier} + \text{M1-M4} \implies \text{one generation} + \text{singlet}. \quad (40)$$

The source ratio does not follow from the gauge group, and a connection alone does not imply a propagating force. The added microscopic models compute a force-generating loop energy only after their Hamiltonians are specified. The synthesis in Theorem 25.1 states the combined result and its exact dependencies. The paper is comprehensive about these interfaces, rather than an assertion that every missing physical principle has been derived.

4.1 Conditional reconstruction statement

Theorem 4.1 (Finite source, phase-compatible interface, and record-frame stabilizer). *Assume the source axioms of Section 7, a seven-state regular marked clock obtained by the declared minimal quadrature rule, and its real doubled cut. Extend the real defect module through $\mathbb{C}_J = \{aI + bJ\}$ after J is selected on the nonfixed source sector. Assume independent standard-form selector mixing, the auxiliary phase register of Section 10.2, and absence of an additional designated right polarization. For smooth carrier bundles assume that a global unit tensor in $\det E_3 \otimes \det E_2$ is retained and patched compatibly. Then:*

- (i) *the source is quaternionic, the recurrence has order seven, and the carriers have complex ranks three and two;*
- (ii) *the source chord response and real doubled-defect gap are $2 \sin(\pi/7)$ and one, respectively; they are not the same kind of spectral separation;*
- (iii) *the retained carrier algebra is $M_3(\mathbb{C}_J) \oplus M_2(\mathbb{C}_J)$, with auxiliary record registers kept separate;*
- (iv) *the unitary stabilizer of the splitting and total tensor is $G_{\text{src}} = S(U(3) \times U(2))$;*
- (v) *connections compatible with this retained tensor have trivial total-determinant holonomy on all loops;*
- (vi) *the common-normalization finite-source response is $r_7 = [2 \sin(\pi/7)]^{-1}$.*

These conclusions concern the declared model. Their selection as physical laws, global tensor descent, and the absence of preferred retained frames are independent assumptions. Full holonomy requires suitable curved connections on a base of dimension at least two and is not an automatic conclusion of the finite clock.

The proofs are distributed through the following sections. Section 5 restates the relevant published completion, real-commutant, and dual-time foundations [7, 8, 9]. Their hypotheses are explicit, and no unpublished article is required to interpret a construction or complete a proof. The source-selection, record-tensor, graph, matter, and Hamiltonian inputs below are additional to those published results.

4.2 Dependency ledger: inputs and conclusions

The entries below distinguish hypotheses from their consequences. An implication is asserted only for the combined inputs stated in its row and in the referenced result. The labels R1–R4, A1–A4, and M1–M4 refer to the assumptions defined in Sections 6, 15 and 24; D1–D4 are unresolved criteria from Section 16, not axioms whose physical validity is claimed.

Table 3: Declared inputs, conditional conclusions, and unresolved dimensional criteria.

Input	Declared content	Conclusion and boundary
Source: Axiom 7.1	Finite-dimensional real unital associative algebra with positive multiplicative quadratic intensity.	Excludes zero divisors; together with noncommutativity, selects $\mathbb{H}$ by Lemma 8.1.
Source: Axiom 7.2	Noncommutative source composition.	Excludes the real and ordinary complex alternatives in the stated associative normed source class.
Source: Axiom 7.3	Exactly one conjugate quadrature per center-free source direction; one reset state; regular orbit without spectators.	Together with the quaternionic source and the explicit clock construction, gives $N = 7$ in Theorem 8.6; minimality and regularity are supplied rules.
Source: Axiom 7.4	Left-selector mixing, retained right commutant, real-cut scalar extension, and an auxiliary product-volume reference.	Gives the declared record construction. The choice of retained reference, its preparation, and its global descent are inputs.
Phase and cut interface	Select J on the nonfixed source sector and extend real doubled defects through $\mathbb{C}_J$.	The complex ranks $3 + 2$ and finite gaps follow in Section 9; the raw cut does not commute with the selected phase.
Global product volume	Retain a global unit tensor in $\det E_3 \otimes \det E_2$ and patch it compatibly.	The stabilizer is $S(U(3) \times U(2))$; total-determinant holonomy is trivial. Local curvature data alone do not establish the global tensor.
R1	Completed predictions use retained record data, with erased selector labels irrelevant and distinct right states retained.	Exact selector forgetting is implemented by the specified channel; it does not erase right–right correlations (Proposition 6.2).

Input	Declared content	Conclusion and boundary
R2	Endpoint structural data include no named internal basis or unrecorded common frame.	Structural endpoint automorphisms act independently. This is a restriction on retained tensors and preparations, not active identification of all carrier states.
R3	Comparators are invertible, complex-linear, norm- and tensor-preserving; they reverse and compose.	R1–R3 and the retained $3 + 2$ tensor data require relational isomorphisms obeying the endpoint gauge law (Theorem 6.6); they do not require nonzero curvature or a gauge-field action.
R4	Short comparisons on an assumed smooth chart admit an identity limit and a first-order expansion.	Adds a compatible connection to the discrete result; neither the chart’s dimension nor autonomous dynamics follows.
A1	Isometric transport preserves the intrinsic marked regular seven-state clock.	Preserves recurrence data while allowing a separate completion environment; it does not derive that environment.
A2	Independent smooth complex isometric embeddings of the triplet and doublet into ranks six and four.	Supplies the environmental modes used by the graph realization.
A3	Source operators, cut, record channel, and product-volume reference transfer consistently; determinant corrections are included when required.	Maintains the declared source and tensor interfaces under the embeddings.
A4	Allows $Z_m = xI_m + yB_m$ with arbitrary anti-Hermitian B_m satisfying $\text{Tr } B_3 + \text{Tr } B_2 = 0$.	A1–A4 give twelve-direction origin-curvature accessibility in Theorem 15.2; they do not establish global realization or propagation.
Invariant amplitude sewing	Haar-average amplitudes at interfaces as in Axiom 12.1.	Projects onto the Gauss sector (Theorem 12.2); density-matrix twirling is a different operation.
Singlet Hamiltonian	Defining-carrier singlet binding, specified unitary hopping, and positive auxiliary penalty.	Exact resolvent, ground-band gap, and frozen-link Hessian follow in Theorems 2.1 and 2.3. The Hamiltonian is supplied, not selected by the source axioms.
Bounded electric dynamics	Finite volume, bounded gauge-invariant electric operator, declared weak-transfer scaling, and dressed-cluster preparation.	The sixth-order remainder and dynamics estimate follow in Theorem 2.6; removing the bound requires (22).
M1	Five CAR creation operations on $E_3 \oplus E_2$, cyclic vacuum, and no independent flavor multiplicity.	Gives the internal module ΛE of rank 32 (Proposition 24.1); CAR is assumed.

Input	Declared content	Conclusion and boundary
M2	Retain the entire degree-zero-connected pair-creation sector and exclude odd internal degree.	With M1, gives $\Lambda^{\text{even}}E$ of rank 16; the finite parity selector does not derive spacetime chirality or mirror decoupling.
M3	Assign each retained component one left-handed Weyl field on an assumed four-dimensional oriented spin spacetime; do not add independent left-handed mirror fields.	Supplies the spacetime interpretation of the internal module. A quaternionic spin lift is a geometric interface, not a proof of this assumption.
M4	Use only the inherited exterior gauge action, without character twists, degree gauge fields, or extra factors.	With M1–M3, yields the one-generation-plus-singlet decomposition, relative hypercharges, and displayed anomaly checks in Theorems 24.3 and 24.5.
D1	Derive completed-event adjacency and weights from the update, without stipulating a cubic lattice.	Unresolved input selection; Proposition 16.1 supplies $\mathbb{Z}^q$ countermodels.
D2	Prove uniform r^3 spatial volume growth and $n^{-3/2}$ diffusion return on expanding finite-volume regimes.	Would establish spatial growth and spectral diagnostics, not by itself a smooth manifold.
D3	Derive a rank-three displacement/coframe limit into $\text{Im } \mathbb{H}$, with a nondegenerate metric and independent spatial and internal frame actions.	Would identify internal source $SO(3)$ with spatial tangent-frame rotations; this identification is not supplied by the clock.
D4	Derive chronology, local causal propagation, and a four-dimensional Lorentzian limit with three-dimensional spatial slices.	Remains independent of a discrete clock or an appended time index.

The exact singlet-loop results have their own Hamiltonian hypotheses and do not require acceptance of SMM/DTT. Conversely, the operational comparison theorem does not require that particular Hamiltonian. The flat-source counterexample (Proposition 6.8) and the dimensional countermodels (Proposition 16.1) prevent these branches from being merged into an unconditional derivation.

5 Published foundations: histories, selected phase, and completion

The three published foundations have distinct scopes. Reference [7] organizes completed-future quotients, compatible complex structures in real commutants, phase-balanced composition, spectral examples, and a cone conditional on supplied length–delay data. Reference [8] isolates the one-particle/operator-algebraic phase selection: an oriented orthogonal flow selects the polar complex structure on its nonzero-frequency sector, without requiring a spectral gap at zero; zero modes and the composite-system rule remain separate. Reference [9] studies coupled temporal evolution and includes an explicit representation of convolution memory with $K(t) = Ce^{tB}D$ by an enlarged local evolution. Its representation is used here as reduced-dynamics motivation,

not as an equivalence between memory and spacetime gauge curvature or a general completeness theorem. The elementary block-elimination identity below is proved directly and does not require the stronger geometric interpretations of that reference. None of these papers establishes the present $3 + 2$ record stabilizer, microscopic singlet-loop Hamiltonian, or chiral Standard-Model dynamics. Those constructions and their extra hypotheses are given explicitly here.

The words *inner* and *outer* designate generative and completed-observable roles. They do not introduce a second Lorentzian time coordinate. A compact recurrence is one possible inner realization; a continuous flow and a finite cyclic clock are different objects until an interpolation or sampling rule is supplied.

5.1 A mathematical dictionary

Term	Definition used in this paper
Generative history	A composable process fragment acting on a state or carrying an amplitude between boundary records.
Completed record	A selected output sufficient for a specified family of future tests; its sufficiency is relative to those tests.
Inner phase	An orthogonal complex structure J selected by an oriented real flow on its nonzero-frequency sector.
Gauge transport	A norm- and tensor-preserving isomorphism between retained fibers, with independent changes of their presentations.
Gauge dynamics	A Hamiltonian or action for physically variable transport and its conjugate variables; this is additional to a frame-change law.
Spatial dimension	A large-scale property of an operational record network and its displacement and diffusion laws, not a carrier rank.
Microscopic completion model	An explicitly specified Hamiltonian and retained sector. Showing that such a model implements an interface is an existence result; selecting it from SMM/DTT is a further problem.

5.2 Behavioral completion and the direction of implication

Let a monoid H act on a set X , let $Q : X \rightarrow R$ be a completion map, and let $\mathcal{O}$ be record tests. A test has the form $x \mapsto o(Q(hx))$, with h in a specified class of admissible future histories. Define

$$x \sim y \iff o(Q(hx)) = o(Q(hy)) \text{ for every admissible } (h, o). \quad (41)$$

The tests can be probabilities rather than deterministic outcomes. Equality then means equality of the complete predicted probability distributions.

Proposition 5.1 (Predictively sufficient completion). *The relation (41) is the coarsest equivalence relation preserving all declared future tests. A proposed record map $q : X \rightarrow Y$ is sufficient for them only if $q(x) = q(y)$ implies $x \sim y$. If the future-history class is closed under right composition by an update k , then $x \sim y$ implies $kx \sim ky$, and that update descends to the behavioral quotient.*

Proof. Each test defines a kernel equivalence relation. Their intersection is an equivalence relation and contains every equivalence preserving all the tests. Sufficiency of q requires every test to factor through it, which gives the stated implication. Finally a test of kx is a test of x with future history hk ; closure makes it one of the tests already included. $\square$

This elementary result is the completion principle of [7] in the form needed below. It does not select Q , the tests, or a Hilbert space. Its force is an exclusion rule: a record that discards a distinction detectable by an admitted future experiment is not complete.

Proposition 5.2 (A group quotient cannot erase its descended inverse). *Suppose a group acts on X and every group element descends through a surjective map q . Every descended transformation is invertible. Consequently noninjectivity of q alone does not produce irreversible quotient dynamics from an action whose inverses also descend.*

Proof. The descended actions satisfy $\bar{h}\bar{h}^{-1} = \bar{e} = id$ and $\overline{h^{-1}}\bar{h} = id$. They are mutual inverses. $\square$

Thus irreversible completion must restrict the admissible updates to a semigroup, fail to preserve inverse descent, discard an environment, or introduce an operational projection. This qualification is essential when using the quotient language: one must not infer irreversibility merely from a many-to-one observation map. For example, the shift on a two-sided sequence is invertible, while forgetting its past allows the forward one-sided shift to descend and prevents its inverse from descending.

5.3 Real phase selection, with its zero-mode boundary

Let $U(s)$ be a real orthogonal representation on a finite-dimensional real Hilbert space $V_{\mathbb{R}}$. Its real commutant is

$$\mathcal{A}_{\mathbb{R}} = \{T \in \text{End}_{\mathbb{R}} V_{\mathbb{R}} : TU(s) = U(s)T \ \forall s\}.$$

A compatible phase is $J \in \mathcal{A}_{\mathbb{R}}$ with $J^* = -J$ and $J^2 = -I$. On the complexification, precisely the operators commuting with its canonical conjugation are complexifications of real operators. This identifies the real commutant with the conjugation-fixed part of the complex commutant. Compatible phases are therefore an intrinsic set of skew-adjoint square roots of $-I$, before any phase is selected [7, 8].

Theorem 5.3 (Polar phase of an oriented real flow). *Let $A^* = -A$ generate $U(t) = e^{tA}$. On $V_{\text{dyn}} = (\ker A)^{\perp}$, put $B = (-A^2)^{1/2}$. Then*

$$J = AB^{-1}, \quad J^2 = -I, \quad J^* = -J, \quad A = JB = BJ.$$

It is the unique orthogonal complex structure in the factorization $A = J|A|$. Reversing A reverses J . A zero mode has no phase selected by this construction.

Proof. A commutes with B by spectral calculus and B is positive and invertible on the stated sector. Hence $J^2 = A^2B^{-2} = -I$ and $J^* = -B^{-1}A = -J$. The factorization determines J uniquely there. At $A = 0$ it gives no information about a square root of $-I$. $\square$

The published flow theorem also handles unbounded skew-adjoint generators and continuous spectrum at zero, with a bounded polar phase on the orthogonal complement of the kernel [8]. The finite proof above suffices for the clock and auxiliary models in this paper. The positive-frequency construction has substantial antecedents in real quantum mechanics [17, 18, 19]; its role here is to fix the order in which real dynamics, phase, and complex transport enter the reconstruction.

The selected J makes $(a + ib)v = av + bJv$. A real orthogonal comparator $T : V_y \rightarrow V_x$ is complex-linear exactly when $TJ_y = J_xT$. Thus complex gauge comparison becomes meaningful after phase selection. That does not require a compact inner circle: many nonperiodic real flows select a J as well.

Proposition 5.4 (Phase does not select an energy scale or spectrum). *For a fixed orthogonal complex structure J , every positive invertible self-adjoint B commuting with J gives a skew-adjoint generator $A = JB$ with the same polar phase J . Conversely every generator with this polar phase has that form. Neither J nor its compatible unitary group fixes B , a spectral gap, a hopping strength, or a gauge coupling.*

Proof. $(JB)^* = -JB$, $-(JB)^2 = B^2$, and hence $|JB| = B$. The converse is the polar factorization. Changing B leaves J unchanged and changes its positive frequencies arbitrarily within the commuting class. $\square$

5.4 Composite systems require phase balance

If $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$ have phases J_V, J_W , their raw real tensor product carries two generally different phase actions. Ordinary complex composition requires the balancing relation

$$(J_V v) \otimes w = v \otimes (J_W w).$$

Equivalently retain the range of the orthogonal projector

$$P_{\text{bal}} = \frac{1}{2}(I - J_V \otimes J_W). \quad (42)$$

Indeed $(J_V \otimes J_W)^2 = I$, and on its -1 eigenspace the two phase actions agree. If the complex dimensions are m, n , this range has real dimension $2mn$, the real dimension of $V_{\mathbb{C}} \otimes_{\mathbb{C}} W_{\mathbb{C}}$. Multiplying the projected elementary real tensor by $\sqrt{2}$ gives the usual product-norm convention. All composite Hilbert spaces below use this selected complex tensor product. This is a composition assumption, not a consequence of the one-particle polar theorem; network tests can depend on it [8, 20].

5.5 Exact reduction: what completion can hide

Let $C : \mathcal{H} \rightarrow \mathcal{H}_{\text{rec}}$ be a coisometry, $CC^* = I$, and write $P = C^*C$. Let $U(t) = e^{-itH/\hbar}$ for a finite-dimensional self-adjoint H . Define the compressed evolution $T(t) = CU(t)C^*$.

Theorem 5.5 (Closure and the missing-record response). *The compressed family is a unitary group for all real t if and only if $[P, H] = 0$. In general,*

$$I - T(t)^*T(t) = CU(t)^*(I - P)U(t)C^*, \quad (43)$$

$$I - T(t)^*T(t) = \frac{t^2}{\hbar^2}CH(I - P)HC^* + O(t^3). \quad (44)$$

Proof. Use $CC^* = I$ and $C^*C = P$ to obtain the first identity. Its quadratic form is the squared norm of $(I - P)U(t)C^*$. If $T(t)$ is unitary for every t , that norm vanishes and $\text{Ran } P$ is invariant under both $U(t)$ and $U(-t)$, hence reducing. Differentiation gives $[P, H] = 0$. Conversely a reducing subspace carries the restricted unitary group. Expansion of the missing amplitude gives $(I - P)U(t)C^* = -it(I - P)HC^*/\hbar + O(t^2)$, which proves the second formula. $\square$

Compression of amplitudes is not automatically a trace-preserving quantum channel. An environment, noise, or a specified measurement instrument is needed when that interpretation is intended. In particular a nonunitary amplitude reduction is not by itself a particle-creation calculation.

Writing a linear evolution in retained and hidden components gives a self-contained memory formula. For constant bounded blocks,

$$\dot{x} = L_{PP}x + L_{PQ}y, \quad \dot{y} = L_{QP}x + L_{QQ}y,$$

variation of constants yields exactly

$$\dot{x}(t) = L_{PP}x(t) + L_{PQ}e^{tL_{QQ}}y(0) + \int_0^t L_{PQ}e^{(t-s)L_{QQ}}L_{QP}x(s) ds. \quad (45)$$

Thus a local enlarged evolution produces a nonlocal reduced equation. This is the elementary block form of projection-based memory methods [21, 22, 23] and connects to the dual-time representation of memory in [9]. Dissipation needs its own stability or bath assumptions; complete positivity is a separate condition [24, 25].

5.6 Three distinct obstructions

Temporal path dependence, retained-fiber curvature, and reduced memory are different quantities. For two supplied temporal connections,

$$\mathcal{F}_{st} = \partial_s \mathcal{A}_t - \partial_t \mathcal{A}_s + [\mathcal{A}_s, \mathcal{A}_t].$$

Vanishing $\mathcal{F}_{st}$ gives local flat transport on a simply connected patch; it does not lower a two-dimensional parameter manifold to one dimension. Nonzero curvature makes contractible loop transport depend on loop geometry, not just on ordinary homotopy class. To obtain a spacetime gauge field from temporal curvature one needs a physical map of record paths into this temporal construction and compatible retained representations. No such map follows just from the use of the word connection.

The memory kernel in (45) can be nonzero in a model with no spatial connection at all. Conversely a smoothly varying protected subspace can have gauge curvature under unitary adiabatic transport. On symplectic one-particle scattering spaces the antilinear part is $\beta = \frac{1}{2}(T + J_{\text{out}}TJ_{\text{in}})$; $\beta = 0$ is phase compatibility, not flatness. Particle-number interpretations additionally require a Fock representation and the usual implementability conditions. These distinctions prevent an irreversibility argument from being mistaken for a derivation of Yang–Mills curvature.

6 Operational assumptions and conditional transport necessity

The relevant necessity is not that every physical description must use redundant potentials. A formulation in terms of relational transport or closed-loop observables may be gauge invariant throughout. The claim is that records with unobservable local internal frames cannot be compared faithfully without transport data equivalent to a gauge connection. This section derives that structure before invoking a differential connection or a gauge-field action.

6.1 What follows from SMM/DTT and what is added

The SMM common-source hypothesis motivates a common compositional origin. DTT distinguishes generative inner evolution from completed outer-time records. These statements do not, by themselves, specify which information the record retains or how different records can be compared. We therefore make their proposed operational completion explicit:

- (R1) **Completed-record sufficiency.** Predictions about completed events depend on their record data, not on a selector label erased by the record map. Distinct right carrier states remain distinct physical data.
- (R2) **No unrecorded frame identification.** The structural data at a record are its Hermitian carrier spaces, selected phase algebra, block identities, and any declared retained tensors. They contain neither a named internal basis nor a prescribed common frame relating distinct records. Local presentations may be selected independently. This principle concerns passive coordinate choices, not active equivalence of all carrier states.

- (R3) **Faithful local comparison.** When neighboring records are compared, the comparison is complex-linear, invertible, preserves the carrier norm and retained structural tensors, has an inverse for reversed comparison, and composes along successive records. It must compare arbitrary carrier amplitudes, rather than only one preferred state.
- (R4) **Smooth correspondence limit.** On an emergent smooth record chart, short comparisons approach identity in a chosen smooth local presentation and admit a first-order expansion. Neither smooth spacetime nor its dimension is derived here.

R1 is implemented for erased labels by the channel of Section 10. R2 is consistent with that channel but is not implied by depolarization alone: a shared externally recorded frame would violate it. R3 is an additional correspondence requirement for reversible comparison. R4 is a continuum assumption. Calling these operational core principles defines the strengthened model; it does not retroactively prove that they follow from the bare ontological postulates.

Theorem 6.1 (Reversible inner steps cannot complete an exact record). *Let $m \geq 2$ and let $\mathcal{E}_m$ be the standard-form selector channel of Definition 10.2, acting on $\mathcal{A}_m = B(V_L \otimes \bar{V}_R)$. If p_a, p_b are distinct orthogonal rank-one projectors on V_L , then*

$$D_{ab} = (p_a - p_b) \otimes I_R \neq 0, \quad \mathcal{E}_m(D_{ab}) = 0.$$

For every reversible step represented on this same algebra by a unitary (or orthogonal, after complexification) conjugation α , including any finite composition of such steps,

$$\|\alpha - \mathcal{E}_m\|_{2 \rightarrow 2} \geq 1.$$

Thus an iteration of reversible inner-clock steps cannot implement exact selector-forgetting completion on this algebra. For the specified mixing semigroup, $\Phi_t^{(m)} - \mathcal{E}_m = e^{-\gamma_m t}(\text{id} - \mathcal{E}_m)$, so exact completion occurs only as $t \rightarrow \infty$, or by taking the record map $\mathcal{E}_m$ itself as an operational projection.

Proof. Definition 10.2 sends both $p_a \otimes I_R$ and $p_b \otimes I_R$ to $I_L/m \otimes I_R$. Normalize D_{ab} to unit Hilbert–Schmidt norm. Any reversible conjugation preserves that norm and remains injective, whereas $\mathcal{E}_m(D_{ab}) = 0$. Hence $\|(\alpha - \mathcal{E}_m)(D_{ab})\|_2 = 1$, proving the bound. The semigroup identity follows from $\mathcal{E}_m^2 = \mathcal{E}_m$. $\square$

This is a limited operational necessity for separating inner generation from completed records. It permits a unitary dilation on a larger system with information transferred to an environment, and it does not derive the particular channel $\mathcal{E}_m$, the seven-state clock, or a physical identification of this separation with DTT. At finite t , $\Phi_t^{(m)}$ is invertible as a *linear* superoperator but is not reversible as a quantum channel: it attenuates D_{ab} , and its inverse is not a positive map on the selector algebra.

Proposition 6.2 (Local record preservation does not erase joint frames). *Let x, y be completed records, with right-carrier algebras $\mathcal{B}_x = B(E_{R_x}) \otimes I$ and $\mathcal{B}_y = I \otimes B(E_{R_y})$ embedded in their joint observable algebra. If a unital completely positive joint channel Ψ fixes $\mathcal{B}_x$ and $\mathcal{B}_y$ pointwise, then it fixes the generated algebra $\mathcal{B}_x \vee \mathcal{B}_y \cong B(E_{R_x} \otimes E_{R_y})$ pointwise. In particular, applying independent instances of $\mathcal{E}_m$ to left selector factors leaves every pre-existing right–right correlation unchanged. Exact local selector erasure therefore does not establish R2 for the joint completed records.*

Proof. For $b \in \mathcal{B}_x$ or $b \in \mathcal{B}_y$, pointwise fixation of its full $*$ -algebra gives $\Psi(b^*b) = b^*b = \Psi(b)^*\Psi(b)$ and likewise for bb^* . Thus b belongs to the multiplicative domain of Ψ . For $b_x \in \mathcal{B}_x$ and $b_y \in \mathcal{B}_y$ it follows that $\Psi(b_x b_y) = \Psi(b_x)\Psi(b_y) = b_x b_y$. Such products span the joint right algebra. $\square$

For example, a prepared right-carrier correlation $|\Omega\rangle\langle\Omega|$, with $|\Omega\rangle = m^{-1/2} \sum_{i=1}^m |i\rangle_x \otimes |i\rangle_y$, is retained by these channels. Its selected pairing of bases is additional joint data; independent endpoint frame automorphisms do not fix that *particular* preparation. This does not invalidate independent passive changes of presentation, nor require that every right-correlated state define a faithful linear comparator. It shows why R2 must specify which *structural tensors and reference preparations* the completion supplies, rather than asserting that local depolarization deletes every possible cross-record reference.

Section 20 supplies one finite joint update and analyzes its admissible preparations. It unifies a selector-completion operation with a carrier–dual return test, but deliberately preserves right–right reference correlations. Independent endpoint structural frames follow only when no cross-record reference has been prepared as structural data; general preparations need not satisfy that condition. Its product-volume coherence is prepared with an explicit reference resource and then preserved. Thus the construction makes several interfaces simultaneous and testable without evading Proposition 6.2 or excluding the flat realization of Proposition 6.8.

Premise	Role in the necessity argument
Common temporal source	Motivation for reconstruction; insufficient to establish independent local frame freedom.
Inner/outer completion	Motivation for a record quotient; requires a concrete choice of retained data.
Exact selector erasure	Makes hidden left resolution labels irrelevant to recorded predictions in the specified model.
Absence of a shared record frame	Supplies independent endpoint frame freedom; independently stated in R2.
Faithful comparison	Excludes a zero or amplitude-destroying comparison; independently stated in R3.
Smooth limit	Turns discrete transport into a connection; stated in R4.
Global product-volume reference	Selects the determinant-linked group; unnecessary for the general transport necessity theorem.

6.2 Future interference forces retention of relational history

The no-shared-frame premise can be complemented by a stronger question: *which history information is completion allowed to forget?* The behavioral criterion of Proposition 5.1 answers this without supposing that all possible shared references are erased.

Let $h_1, h_2 : y \rightarrow x$ be two histories with identical endpoint record structures, and let $U_1, U_2 : E_y \rightarrow E_x$ be their unitary transports. Assume a coherent comparison test can prepare $\psi \in E_y$, place the two histories in separate control paths, and recombine them. Its plus-port probability is

$$p_+(\psi, \phi) = \frac{1}{2} \left[1 + \operatorname{Re} \left(e^{i\phi} \langle \psi, U_2^* U_1 \psi \rangle \right) \right]. \quad (46)$$

The control paths and their relative phase are declared physical reference data. They are not produced by a frame relabeling.

Theorem 6.3 (Endpoint-only completion fails for distinguishable transport). *Suppose the admitted future tests include the coherent comparisons (46), with arbitrary normalized input and a calibrated reference history. If $U_1 \neq U_2$, a predictively sufficient record cannot identify the two histories solely because their endpoint structural records agree. The record must retain their transport distinction or relational data sufficient to reconstruct all its admitted interference statistics. In local frames this retained information transforms by the endpoint gauge-link law.*

Proof. The unitary $W = U_2^* U_1$ differs from I . It therefore has an eigenvector ψ with eigenvalue $e^{i\theta} \neq 1$. At $\phi = 0$ the comparison probability is $(1 + \cos \theta)/2 < 1$, whereas the comparison of U_2 with itself gives one. Hence a declared future test separates the histories, and Proposition 5.1 forbids identifying them in a sufficient record. For independent changes of endpoint frame, $U'_i = g_x^{-1} U_i g_y$, so $W' = g_y^{-1} W g_y$ and the probability is unchanged when $\psi' = g_y^{-1} \psi$. $\square$

This is a necessity of retained relational information, not a necessity of storing a literal microscopic trajectory. An endpoint record enlarged by its transport class is sufficient for these tests. A prepared shared frame can supply that class and must then be counted as part of the record. Conversely, if coherent path references are excluded, tests may identify a central phase or a larger class of transports. The operational test family determines the quotient; a universal ability to resolve every phase is not assumed silently.

Corollary 6.4 (A conditional route from histories to a connection). *Assume retained comparison histories compose, invert under path reversal, are invariant under reparametrization and thin homotopies, and depend smoothly and locally on paths. If their action on retained carriers is faithful, complex-linear, and tensor preserving, they supply smooth gauge transport. Their short-link limit is a compatible connection, and a distinguishable loop gives nontrivial holonomy.*

Proof. Composition and reversal give a path transport functor. In a local trivialization its short-link derivative defines A_μ as in Theorem 6.6; subdivision and smoothness recover its path-ordered transport. Thin homotopies remove changes that sweep out no area. The gauge law follows from the independent endpoint frames. $\square$

The general transport-functor/connection correspondence is established geometry [26]. The application here places the necessity one step earlier: future-test sufficiency disallows deleting observable history transport. A single nontrivial loop can still be a flat topological holonomy. Local curvature requires a nontrivial area-order response of arbitrarily small contractible loops.

6.3 Local structure does not select a comparator

For one record let $E = E_3 \oplus E_2$ with its Hermitian metric, phase unit, splitting, and total unit volume tensor. Its adapted frames form a torsor for $G_{\text{src}} = S(U(3) \times U(2))$: the group acts freely and transitively on compatible frames, with no distinguished element. Two records have independent torsors until relational data are supplied.

Theorem 6.5 (No structural-data-only faithful comparison). *There is no nonzero block-preserving linear comparator $C_{xy} : E_y \rightarrow E_x$ constructed naturally from only the separate structural data of two records, if their adapted frame automorphisms act independently. In particular no such construction can provide the faithful comparison required by R3.*

Proof. Naturality under endpoint automorphisms, which fix the structural input, requires

$$C_{xy} = g_x^{-1} C_{xy} g_y \quad \text{for all } g_x, g_y \in G_{\text{src}}.$$

Take $g_y = I$. On the triplet choose $g_x = (\zeta I_3, I_2)$ with $\zeta = e^{2\pi i/3}$; it belongs to G_{src} and forces the triplet block to vanish. On the doublet choose $g_x = (I_3, -I_2)$, also in G_{src} ; it forces the doublet block to vanish. Thus $C_{xy} = 0$. The same argument applies to ordinary $U(m)$ carriers whenever their automorphism group has no fixed vector in the defining representation. $\square$

The theorem does not deny a comparator supplied by a common ambient source, an overlap, or a recorded reference frame. Such objects are additional relational data and are precisely what separate endpoint structural data lack. Nor does it say that passive frame freedom makes all active states physically equivalent.

Theorem 6.6 (Gauge transport is the minimal faithful comparison structure). *Under R1–R3 for the retained $3 + 2$ carrier with a global total-volume tensor, a faithful comparison requires additional relational isomorphisms between retained fibers. Their matrices in local adapted frames are G_{src} -valued links C_{xy} , with*

$$C'_{xy} = g_x^{-1} C_{xy} g_y, \quad C_{yx} = C_{xy}^{-1}. \quad (47)$$

Their products compare successive records; closed products are defined up to conjugation. Under R4, short links give a compatible connection with the usual inhomogeneous transformation law. No connection, covariant derivative, or Yang–Mills action is assumed in R1–R3. Here “minimal” means that endpoint data must be supplemented by relational comparison; it is not a theorem of uniqueness or categorical minimality among all possible implementations.

Proof. The preceding theorem excludes a faithful comparator from structural endpoint data alone. A norm-preserving complex-linear isomorphism is unitary by the polarization identity; preservation of the splitting and total tensor makes it a G_{src} isomorphism. Writing its intrinsic action $E_y \rightarrow E_x$ in independently changed frames gives (47). Composition cancels intermediate frames, and a closed product transforms by conjugation at its starting point. Under R4 write

$$C_{x, x+\epsilon e_\mu} = I + \epsilon A_\mu(x) + o(\epsilon).$$

Expanding $g_x^{-1} C_{x, x+\epsilon e_\mu} g_{x+\epsilon e_\mu}$ gives

$$A'_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g.$$

Unitarity makes A_μ anti-Hermitian, and tensor preservation imposes the total-trace constraint. The differential comparison is $D_\mu = \partial_\mu + A_\mu$, derived from the link expansion. Its Leibniz rule follows from differentiating scalar multiplication rather than being postulated to define a connection. $\square$

If the global total-volume tensor is not retained, the same necessity argument uses the automorphism group of the actual declared record structure instead of G_{src} . The particular determinant-linked group depends on the additional tensor-retention premise. This establishes necessity of gauge-covariant transport within the operational completion of SMM/DTT. It establishes neither a unique transport law nor independent gauge-field degrees of freedom. Closed-loop formulations encode the same comparison content without introducing local potentials as observable entities. Relational reference-frame changes already have a substantial literature [4]; the application here is to this paper’s explicit completion and record model.

6.4 Which comparisons have observable content?

Proposition 6.7 (Tree reduction and cycle observables). *On a finite connected record graph with inverse links, a choice of frames can make every spanning-tree link identity. The remaining edges give $|\mathcal{E}| - |\mathcal{V}| + 1$ fundamental cycle comparators, defined up to simultaneous conjugation at a root. Thus local link matrices are gauge dependent, while cycle spectra and trace deficits carry frame-independent comparison information.*

Proof. Fix a root frame. For a tree edge directed from parent x to child y , choose $g_y = C_{xy}^{-1} g_x$. Its transformed backward comparator is identity. Recursion defines all frames uniquely once the root is chosen. Every non-tree edge closes one fundamental cycle; changing the root frame conjugates all such products together. $\square$

A tree has no independent loop holonomy, and a one-dimensional smooth chart has zero local curvature. Inner-time compactness may support a global loop but does not force a nonzero spacetime curvature two-form. Curvature records nontrivial local cycles in the smooth comparison geometry; its existence requires more than unobservable labels.

6.5 A sharp limit: the bare source permits flat transport

Proposition 6.8 (Flat realization of the bare reconstruction assumptions). *The finite source, phase-compatible defect, selector channel, and retained product tensor admit a realization with constant carrier projectors, identity comparison links, zero curvature, and no autonomous gauge field. Therefore those bare assumptions do not entail interacting Yang–Mills dynamics.*

Proof. Take the exact seven-state source and its scalar-extended defect, identically at every point of a trivial record chart. Use the same standard-form and phase-register channels pointwise, with a fixed total-volume reference. Set $C_{xy} = I$ in that fixed trivialization. All finite source and record identities hold, the connection is flat, and no kinetic gauge field has been introduced. Passive local changes of presentation produce pure-gauge matrices but do not alter the zero curvature. $\square$

The distinction is logically useful: R2–R3 make relational transport necessary, but nonzero curvature and autonomous dynamics require further physical input. Any stronger claimed derivation from the same assumptions would have to exclude this explicit flat realization.

6.6 Five logically distinct reconstruction steps

The necessity argument is most transparent when its targets are separated. Every implication in the following table is proved in this manuscript; none licenses replacing the assumptions of a later row by the conclusion of an earlier one.

Target	Necessary declared input	Proved conclusion	Boundary or counterexample
Predictive retention	Specified coherent future tests and sufficient completion	Transport distinctions resolved by those tests survive	Removing phase-sensitive tests can identify central phases.
Comparison covariance	Independent endpoint presentations, faithful complex-linear norm/tensor-preserving comparison	Endpoint link law; connection only with a smooth path limit	A supplied common frame is extra relational data; endpoint structures alone give zero comparator.
Internal group	Specified 3 + 2 modules, selected scalar phase, splitting, and retained parallel product-volume tensor	Stabilizer $S(U(3) \times U(2))$	No volume gives $U(3) \times U(2)$; separate volumes give $SU(3) \times SU(2)$.
Gauss sector	Coherent invariant amplitude sewing and neutral internal boundary condition	Projector onto invariant amplitudes	Twirled nonsinglet density matrices remain possible without sewing.
Gauge dynamics	Quantum links, hopping, completion gap, and independent electric motion	Static loop energy; restricted controlled effective Hamiltonian	Flat prescribed comparisons have no autonomous field; zero hopping removes this generated interaction.

The first two rows are generic quantum-information and transport statements. SMM/DTT supplies their proposed generative-history and completion interpretation, not their exclusive

microscopic realization. The third row depends on the paper’s source and retained-tensor choices; the last two rows require physical composition and Hamiltonian inputs. Section 20 implements several of these interfaces in one update but does not select all of them from the temporal postulates.

The observable algebra $M_3 \oplus M_2$ must not be substituted for its carrier modules. Conjugation on this algebra annihilates both scalar block phases, so it is not a faithful action of the determinant-linked group. The defining module $E_3 \oplus E_2$ is faithful; phase-sensitive comparison tests and the retained volume tensor expose its central circle. All statements involving the relative $U(1)$ retain those modules and reference tensors explicitly. A record restricted to adjoint observables alone has a different operational quotient.

6.7 The scope of a necessity claim

Theorems 6.5 and 6.6 concern the data needed to compare independently unframed records. They do not say that SMM/DTT is required in every microscopic realization of gauge theory. Conventional lattice gauge theory starts from link variables and a specified local gauge symmetry [11, 12]; other many-body mechanisms can produce emergent gauge excitations [16]. These are logically distinct starting points.

A stronger SMM/DTT necessity statement would have to name a class $\mathfrak{C}$ of microscopic update rules and an independently specified target $\mathfrak{T}$: a record network with a three-dimensional spatial scaling regime, relational G_{src} comparisons, and autonomous gauge response. The criteria defining $\mathfrak{C}$ must be motivated without building inner time, a seven-state clock, three spatial axes, the gauge group, or their desired conclusions into its definition. One would then prove that every member of $\mathfrak{C}$ realizing $\mathfrak{T}$ contains an identifiable inner generation/completion distinction and the requisite record quotient, or else produce a counterexample. Such a statement would establish conditional uniqueness within $\mathfrak{C}$, not an unrestricted ontological necessity. In particular, a counterfactual rule that omits completion must be analyzed with the same observables and low-energy criteria as the original rule. A model whose every link is the identity demonstrates insufficiency of the current axioms, but the existence of a flat *configuration* or vacuum does not disqualify a dynamical gauge theory.

7 Minimal source axioms

Axiom 7.1 (Compositional source). *Source amplitudes form a finite-dimensional real unital associative algebra. Composition is bilinear and closed, and a positive quadratic intensity is exactly multiplicative.*

Axiom 7.2 (Nonclassicality). *Source composition is noncommutative. This excludes the scalar and ordinary complex alternatives before any recurrence order is considered.*

Axiom 7.3 (Minimal reversible completion). *Every center-free real source direction receives exactly one conjugate quadrature. A single reset state closes the orbit. The reset orbit is regular and there are no inaccessible, duplicated, or spectator states.*

Axiom 7.4 (Canonical open record and full-event completion). *Microscopic resolution data act in the left standard representation and are mixed primitively; retained carrier data are the right commutant. The completion first forms real cut defects and then extends scalars through the selected phase algebra. A separate auxiliary tensor register retains the neutral product volume coherence and mixes the relative block phase. The physical choice to retain that coherence is an assumption. The record map is not a closed automorphism and does not act as matter-state decoherence.*

Remark 7.5 (Physical assumptions and target dependence). The source axioms do not explicitly name a gauge group, but absence of its name is not a proof of target independence. Associativity, one quadrature pair per source direction, a regular reset orbit, the doubled defect interface, the standard-form separation, and retention of only the total block volume are substantive restrictions. In particular, the total-volume retention rule uses the already constructed $3 + 2$ ranks and specifies a tensor category whose stabilizer is the stated group. Their independent motivation and robustness under alternative record mechanisms are part of the research program.

7.1 Ontological interpretation and mathematical content

SMM motivates a common compositional source rather than independent primitive spatial entities. DTT motivates a distinction between inner reconstruction and completed outer-time records. Neither statement alone selects a division algebra, a seven-state clock, a complex scalar field, a smooth spacetime, or local gauge freedom. The model supplies explicit additional assumptions at each interface. Complex phases here use a selected J with $J^2 = -I$; they must not be identified with a split-complex temporal unit satisfying $j^2 = +1$.

The distinction between a common origin and a common observable frame is central. A microscopic marker may fix a frame before recording, while the record need not retain that frame. The construction below separates microscopic selector labels from retained state spaces. A local gauge description additionally requires independently chosen charts over a smooth record base; a finite source at a single point cannot itself establish locality.

8 Quaternionic source rigidity and order selection

Lemma 8.1 (Frobenius source classification). *Under Axioms 7.1 and 7.2, $A \cong \mathbb{H}$.*

Proof. Positive norm multiplicativity excludes zero divisors. Finite dimensionality and associativity imply that every nonzero element is invertible. Frobenius' theorem leaves only $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$; noncommutativity selects $\mathbb{H}$. $\square$

Corollary 8.2 (Intrinsic displacement module). *The center-free module*

$$W = A/Z(A) \cong \text{Im } \mathbb{H}$$

has real dimension three. The norm induces its Euclidean metric. Quaternion multiplication gives the nonzero alternating form

$$\omega(u, v, w) = \langle uv, w \rangle,$$

which fixes an orientation. The connected inner automorphism group is $SO(W)$.

Lemma 8.3 (Quadrature lower bound). *A nontrivial oriented real frequency mode has real dimension at least two. Therefore a reversible realization of all d independent center-free source directions, with no fixed non-source line, has dimension at least $2d$ away from the marker.*

Proof. A one-dimensional real orthogonal action has eigenvalue $+1$ or -1 and carries no continuously oriented phase. A nonreal unit-modulus eigenvalue occurs with its complex conjugate and hence contributes a real two-plane. Completeness and no spectators require exactly one such plane per source direction. $\square$

Definition 8.4 (Explicit microscopic clock). *For an oriented orthonormal source frame $F = (f_1, \dots, f_d)$, let*

$$\mathcal{H}_F = \mathbb{R}u_0 \oplus W_c \oplus W_s, \quad N = 1 + 2d.$$

Let $c(f_k)$ and $s(f_k)$ be the two copies of f_k . Define

$$\begin{aligned} K_F u_0 &= u_0, \\ K_F c(f_k) &= \cos(2\pi k/N) c(f_k) + \sin(2\pi k/N) s(f_k), \\ K_F s(f_k) &= -\sin(2\pi k/N) c(f_k) + \cos(2\pi k/N) s(f_k), \end{aligned}$$

and

$$\Omega_F = \frac{1}{\sqrt{N}} u_0 + \sqrt{\frac{2}{N}} \sum_{k=1}^d c(f_k).$$

The reset channel is $\mathcal{C}_F(\rho) = |\Omega_F\rangle\langle\Omega_F| \text{Tr } \rho$.

Lemma 8.5 (Regularity and uniqueness of the reset vector). *The vectors $e_n = K_F^n \Omega_F$, $n \in \mathbb{Z}_N$, form an orthonormal basis. Conversely, for the simple spectrum of K_F , a cyclic vector has an orthonormal orbit exactly when every spectral weight has modulus $N^{-1/2}$. Reflection and source orientation fix the remaining phase convention up to an orientation-preserving source frame.*

Proof. The inner product $\langle e_m, e_n \rangle$ is the inverse discrete Fourier transform of the uniform spectral measure on all N th roots of unity, hence equals δ_{mn} . Conversely, orthogonality of all orbit translates says that the Fourier transform of the spectral weights is the Kronecker delta, so the weights are uniform. Reality pairs conjugate phases; the marked reflection fixes cosine phases; source orientation chooses between K_F and K_F^{-1} . $\square$

Theorem 8.6 (Unique order theorem). *The source of Lemma 8.1 has $d = 3$. The clock of Definition 8.4 therefore has $N = 1 + 2d = 7$, and Lemma 8.5 makes its first-return order seven.*

Proof. By Corollary 8.2, $d = 3$. Minimal quadrature gives dimension $1 + 2d = 7$. A regular orbit is an orthonormal basis, so its length equals the Hilbert-space dimension. The explicit eigenvalues are all seventh roots, and no proper positive power fixes them all. $\square$

Proposition 8.7 (Exclusion of nearby orders). *Within the axioms:*

- (a) $N = 3, 5$ provide only one or two nontrivial conjugate planes and cannot read out all of W ;
- (b) $N = 9, 11, \dots$ contain at least one additional Fourier plane and violate no-spectator minimality;
- (c) even N contain a real -1 character, which is not a conjugate quadrature plane and violates the all-nonzero-modes phase rule.

No-go theorem 8.8 (Octonionic fork). *If associativity is removed while positive norm composition is retained, the octonions become admissible. Their center-free imaginary module has rank seven and the same minimal-dilation rule gives order fifteen. This is an alternative model branch. Dropping associativity merely admits it; it does not force an octonionic source or by itself provide an empirical falsifier.*

9 Carrier, complex structure, defects, and gaps

Let $Re_n = e_{-n}$. Then $R^2 = 1$, $R\Omega = \Omega$, and $RKR = K^{-1}$.

Lemma 9.1 (Odd carrier and physical isotropy). *The reflection-odd subspace has dimension three and is canonically the sine-quadrature copy W_s . After the frame-selector diffusion has forgotten F , the only retained positive bilinear form on this output is a scalar multiple of the quaternionic norm, and the only retained orientation is the quaternionic orientation. Hence the retained output metric has isotropy group $SO(3)$. This is not yet a construction of three-dimensional spacetime.*

Proof. Reflection is $+1$ on the cosine copy and -1 on the sine copy. The oriented-frame selector is ergodic under $SO(W)$, whose defining representation on W is irreducible. Schur's lemma makes every invariant symmetric bilinear form scalar. The source three-form fixes orientation. $\square$

For the complexified regular clock define

$$P_k = \frac{1}{7} \sum_{n=0}^6 e^{-2\pi i k n / 7} K^n, \quad z_k = \sqrt{7} P_k \Omega, \quad k = 1, 2, 3.$$

These are the positive-frequency projectors and marker-calibrated vectors. Complexification here is a device for displaying the spectral lines; the physical complex scalar structure is then identified through the selected polar unit.

Lemma 9.2 (Positive-frequency complex structure). *On the nonfixed sector, let $A = \log K$ be the oriented principal skew-adjoint logarithm. Its polar phase*

$$J = A|A|^{-1}$$

is the unique orthogonal complex structure satisfying $A = J|A|$. The positive-frequency lines $\mathbb{C}z_k$, $k = 1, 2, 3$, form $E_3 \cong \mathbb{C}^3$.

9.1 The cut obstruction and a phase-compatible completion functor

Proposition 9.3 (Raw marked cuts do not preserve the source phase). *Let J be the polar phase on the nonfixed real clock sector, extended by zero on the fixed mode, and $P_\Omega = I - |\Omega\rangle\langle\Omega|$. Then $[P_\Omega, J] \neq 0$. For the regular seven-state source,*

$$\|[P_\Omega, J]\| = \sqrt{6/7}.$$

Consequently the raw cut cannot be declared complex-linear over the original polar structure.

Proof. The marked vector has weight $1/7$ on the fixed mode and $6/7$ on the nonfixed modes. Put $v = J\Omega$. Skew-adjointness gives $\langle\Omega, v\rangle = 0$, and $\|v\|^2 = 6/7$. Direct calculation gives $[P_\Omega, J] = |\Omega\rangle\langle v| + |v\rangle\langle\Omega|$, whose nonzero eigenvalues are $\pm\|v\|$. $\square$

The obstruction is exact. It cannot be removed by simply renaming the ordinary scalar i as the selected phase J .

Definition 9.4 (Scalar-extension completion rule). *On the nonfixed source space define the real algebra*

$$\mathbb{C}_J = \{aI + bJ : a, b \in \mathbb{R}\}, \quad J^2 = -I.$$

This is a copy of the complex field with distinguished oriented imaginary unit J . For every real interface module D , define

$$\mathcal{S}_J(D) = D \otimes_{\mathbb{R}} \mathbb{C}_J.$$

For a real map $L : D \rightarrow D'$ set $\mathcal{S}_J(L) = L \otimes 1$. Scalar multiplication acts on the second factor. The norm is obtained from the real metric and the usual norm $a^2 + b^2$ on the phase algebra.

This is a specified composition rule, not a claim that every microscopic defect is already a complex subspace of the original clock. Its merit is precise compatibility: it uses the already selected phase algebra without requiring $LJ = JL$ on a real domain where such a condition is inappropriate. Under source conjugacy $J' = UJU^{-1}$ the map $aI + bJ \mapsto aI + bJ'$ canonically identifies the scalar algebras; reversal of the selected orientation conjugates them.

Theorem 9.5 (Phase-compatible doubled defect). *Let $V_\Omega = \Omega^\perp$ over $\mathbb{R}$ and*

$$T_R = P_\Omega K P_\Omega|_{V_\Omega}, \quad D_R = \begin{pmatrix} 0 & T_R^* \\ T_R & 0 \end{pmatrix} \quad \text{on } V_\Omega \oplus V_\Omega.$$

Here T_R^* is the real adjoint and the doubled space has its direct-sum real inner product. Then $\ker D_R$ has real dimension two and the nonzero singular gap is one. The extended operator $D_J = D_R \otimes 1$ is $\mathbb{C}_J$ -linear and self-adjoint, and its kernel

$$E_2 = \ker D_R \otimes_{\mathbb{R}} \mathbb{C}_J$$

has complex dimension two with the same nonzero gap.

Proof. In the orbit basis the real cut is the length-six truncated unilateral shift. Its singular values are five ones and one zero. Thus the doubled kernel is

$$\mathbb{R}(K^{-1}\Omega, 0) \oplus \mathbb{R}(0, K\Omega).$$

Tensor extension preserves these finite-matrix kernel and singular-value statements, while scalar multiplication commutes with $D_R \otimes 1$ by construction. $\square$

The in/out vectors belong to different doubled summands. Their microscopic labeling need not define a retained right polarization. The physical retention of E_2 and the scalar-extension rule remain explicit interface assumptions. The result repairs compatibility; it does not derive that physical interface from the source algebra alone.

Theorem 9.6 (Finite spectral protection). *Unitary transport of the exact source pair preserves the chord response $\Delta_3 = 2\sin(\pi/7)$ and the scalar-extended doubled defect gap $\Delta_2 = 1$. The defect zero cluster is isolated by distance one. The positive-frequency cluster of $-\mathrm{i}\log K$ has rank three and is separated from zero by $2\pi/7$. For this Hermitian generator and for D_J , self-adjoint perturbations smaller than half their respective cluster-separating distances preserve the corresponding ranks. More generally a Riesz rank persists when the resolvent remains invertible on a fixed enclosing contour.*

Remark 9.7. The chord response is not the positive-frequency Riesz separation: $|I - K|$ has conjugate-pair degeneracies and by itself does not select the complex triplet. Protection of a whole cluster also does not imply degeneracy of all internal eigenvalues. A frame change is covariant even when its matrix generator is nondegenerate; arbitrary $U(3)$ matrices need not commute with the fixed clock generator.

10 Record standard form and selector-forgetting theorem

10.1 Why a copied selector is not enough

No-go theorem 10.1 (Projector-erasure obstruction). *Let $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{B}$ be a conditional expectation that fixes $\mathcal{B}$ pointwise. If a microscopic line projector p belongs to $\mathcal{B}$, then $\mathcal{E}(p) = p$ and it is not forgotten. Therefore one cannot erase the projectors that select a multiplet while preserving the full algebra containing those same projectors pointwise.*

One explicit resolution is standard form: selectors and carriers are commuting left and right actions of the same matrix algebra.

Definition 10.2 (Canonical standard-form record channel). *For $V \cong \mathbb{C}^m$, identify*

$$L^2(M_m, \mathrm{Tr}) \cong V_L \otimes \bar{V}_R.$$

Let M_m act on the left by L_a and its commutant act on the right by R_b . For $\gamma_m > 0$ and $t \geq 0$, define

$$\mathcal{E}_m(X) = \frac{I_L}{m} \otimes \mathrm{Tr}_L(X), \quad \Phi_t^{(m)} = \mathcal{E}_m + e^{-\gamma_m t}(\mathrm{id} - \mathcal{E}_m).$$

Theorem 10.3 (Exact selector forgetting with carrier preservation). *For $m = 3, 2$, $\gamma_m > 0$, and each $t > 0$:*

$$\begin{aligned}\text{Fix}(\Phi_t^{(m)}) &= I_L \otimes M_m(\mathbb{C})_R, \\ \mathcal{E}_m(p_a \otimes I_R) &= \frac{I_L}{m} \otimes I_R, \\ \mathcal{E}_m(I_L \otimes b) &= I_L \otimes b.\end{aligned}$$

At $t = 0$, $\Phi_0^{(m)} = \text{id}$ and its fixed algebra is the full algebra. The right algebra is the common fixed algebra of the semigroup and the fixed algebra of every positive-time map. The convergence estimate is exact for all $t \geq 0$:

$$\|\Phi_t^{(m)} - \mathcal{E}_m\|_{2 \rightarrow 2} = e^{-\gamma_m t}.$$

After positive 3–2 block dephasing, the full fixed algebra is

$$M_3(\mathbb{C})_R \oplus M_2(\mathbb{C})_R.$$

Proof. $\mathcal{E}_m$ is the Hilbert–Schmidt orthogonal projection onto the right commutant, hence is unital, trace preserving, completely positive, and idempotent. The displayed semigroup follows from $\mathcal{E}_m^2 = \mathcal{E}_m$. Its fixed space is exactly the range of $\mathcal{E}_m$. Block dephasing removes off-diagonal intertwiners. $\square$

Corollary 10.4 (No designated retained in/out polarization). *The two designated left defect projectors have the same record image. Therefore the left selector does not supply a preferred polarization of the right carrier. The full right matrix algebra nevertheless contains rank-one projectors and self-adjoint involutions: these observables are not erased. The right carrier has frame group $U(2)$ only if no additional retained tensor, Hamiltonian constraint, or record coupling singles out a right polarization.*

Remark 10.5 (Passive frames versus physical transformations). A passive frame change transforms states, observables, and generators together; it is not a transformation of the state alone while every observable remains fixed. Erasing a left selector does not identify all distinct right states or make every right unitary physically trivial. The record channel is compatible with right frame changes, but local frame redundancy is a separate representational principle. Also, Φ_t is primitive on the selector factor, not on the entire standard-form algebra, whose right fixed algebra is nontrivial.

10.2 An explicit auxiliary phase register

Let $\mathcal{R} = \mathbb{C}^4$ with ordered basis $|00\rangle, |10\rangle, |01\rangle, |11\rangle$. These denote auxiliary volume-label occupations, not physical carrier states. Define

$$Q_{\mathcal{R}} = \text{diag}(0, -6, 6, 0), \quad \mathcal{L}_{\phi}(X) = -\frac{\gamma_{\phi}}{2}[Q_{\mathcal{R}}, [Q_{\mathcal{R}}, X]], \quad \gamma_{\phi} > 0.$$

Here the charges are the determinants of the primitive relative module weights -2 and 3 , multiplied by block ranks three and two.

Theorem 10.6 (Product-volume coherence without carrier decoherence). $\Xi_t = e^{t\mathcal{L}_{\phi}}$ is a unital trace-preserving completely positive semigroup. For matrix units,

$$\Xi_t(|a\rangle\langle b|) = e^{-\gamma_{\phi}t(q_a - q_b)^2/2} |a\rangle\langle b|.$$

The total-volume coherence $C_{\text{tot}} = |11\rangle\langle 00|$ is fixed, whereas the separate volume coherences $C_3 = |10\rangle\langle 00|$ and $C_2 = |01\rangle\langle 00|$ decay as $e^{-18\gamma_{\phi}t}$. The register fixed algebra is $M_2 \oplus \mathbb{C} \oplus \mathbb{C}$, of complex dimension six.

Proof. The generator equals $\gamma_\phi(QXQ - \{Q^2, X\}/2)$ and is of Lindblad form. Alternatively, Ξ_t is an average of conjugations $e^{i\alpha Q} X e^{-i\alpha Q}$ with centered Gaussian variance $\gamma_\phi t$. Its action on matrix units and fixed charge eigenspaces gives all assertions. $\square$

For each standard-form block take the tensor-product channel $\Phi_t^{(m)} \otimes \Xi_t$; inter-block coherences can be removed by an independent block-dephasing generator. This preserves $I_L \otimes b_R \otimes I_R$ for every physical right observable. The auxiliary fixed algebra is real mathematical data and is not silently deleted from the full channel. Its identification as a label register rather than a propagating matter sector is a modeling assumption. Therefore the carrier fixed algebra alone has dimension $9 + 4 = 13$, while including this register gives $(9 + 4)6 = 78$.

To attach the register to volume tensors, use $|10\rangle$ and $|01\rangle$ as labels for the two block wedges and $|11\rangle$ for their product. A prepared nonzero C_{tot} retains a product-phase reference. The channel preserves but does not create that reference. No right carrier state is averaged over the relative phase. This explicitly separates label forgetting from physical charge decoherence.

Proposition 10.7 (Primitive phase-calibration descent). *On the exterior completion space let N_3 and N_2 count triplet and doublet occupancy. The unique primitive nonzero block-scalar integer generator annihilating the fully occupied bidegree $(3, 2)$ is, up to sign,*

$$Q_{\text{rel}} = -2N_3 + 3N_2.$$

Its passive phase action on frame tensors multiplies the triplet and doublet defining modules by $e^{-2i\alpha}$ and $e^{3i\alpha}$. The separate top wedges have charges -6 and $+6$, while their product and the vacuum-to-full-completion operator have charge zero. Averaging this phase in an auxiliary frame-label register forgets both separate determinant frames while preserving their product. This averaging is not a channel on physical carrier states: applying it there would generally remove charged coherences and could conflict with preservation of the full right matrix algebra. The channel above realizes the tensor-label interface; its physical identification and prepared reference remain part of the record hypothesis.

Proposition 10.8 (Parity-output factorization). *The spatial output carries the source $SO(3)$ action; the retained internal carriers carry independent right $U(3) \times U(2)$ actions. There is no nonzero invariant tensor in*

$$W_{\text{sp}}^* \otimes \text{End}(E_3) \quad \text{or} \quad W_{\text{sp}}^* \otimes \text{End}(E_2)$$

under the full product action. Thus there is no invariant mixed tensor if independent spatial and internal frame actions are retained as a product. Establishing that independence from a microscopic joint channel is an additional model condition.

Proof. The conjugation invariants of $\text{End}(E_m)$ under $U(m)$ are the scalars. The defining $SO(3)$ module has no invariant linear functional. The invariant tensor space therefore vanishes. $\square$

11 Canonical determinant tensor, global group, and phase atlas

Lemma 11.1 (Marker-fixed spectral phases). *For a regular marked cycle, $\|P_k \Omega\| = 1/\sqrt{7}$. Hence $z_k = \sqrt{7}P_k \Omega$ is a canonical unit vector in the k th positive-frequency line; no arbitrary eigenvector phase remains. Likewise $K^{\pm 1} \Omega$ canonically orient the two defect lines.*

Locally let e_3 be a unit wedge of the positive-frequency right carrier and let e_2 be the wedge of the two scalar-extended real defect vectors, transferred to the right module. Then $\tau_{\text{src}} = e_3 \otimes e_2$ is a local unit tensor. Global descent of these products is a separate assumption: neither a local marker nor the four-state register determines all transition functions. On a trivial record chart the explicit construction below supplies a global reference.

Theorem 11.2 (Canonical total determinant cancellation). *Let τ_{src} be a stipulated globally retained unit tensor in $\det E_3 \otimes \det E_2$, represented locally by the product-volume reference. It survives the declared record channel, although no individual spectral or defect line and no separate block volume is retained as a named tensor. Its unitary stabilizer is*

$$S(U(3) \times U(2)) = \{(u_3, u_2) : \det u_3 \det u_2 = 1\}.$$

Compatibility here includes $\nabla_{\tau_{\text{src}}} = 0$; it is a restriction on allowed transport, not an automatic property of any spectral connection. Consequently total determinant holonomy is trivial on every loop, including torsion and flat-holonomy loops.

Proof. In an exact smooth family the spectral-projector formula and marked defect vectors provide nonvanishing local tensors. The global assertion assumes that their product patches as the retained tensor specified by the total-volume retention rule; local marker-fixed phases alone do not prove this gluing. A product wedge is unchanged by determinant-one internal frame changes but does not retain a named basis. Its declared reference retains the product phase. By Proposition 10.7, the two separate block wedges are charged and averaged away, whereas their product is neutral. A block unitary acts on the product wedge by $\det u_3 \det u_2$, giving the stabilizer. Parallelity of a global unit tensor makes its holonomy exactly one; this is stronger than zero curvature. $\square$

Corollary 11.3 (Global quotient and unique relative Abelian factor). *There is a canonical isomorphism*

$$S(U(3) \times U(2)) \cong \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6},$$

induced by

$$(g_3, g_2, z) \mapsto (z^{-2}g_3, z^3g_2).$$

The kernel has six elements. Because the selector fixed algebras are scalar and the total determinant removes one of the two block-center phases, exactly one Abelian factor survives.

Proposition 11.4 (Smooth compact phase atlas). *Let $(g_{3,ab}, g_{2,ab})$ be source-compatible frame transitions. Then*

$$h_{ab} = \det g_{2,ab} = (\det g_{3,ab})^{-1}$$

is a smooth $U(1)$ Čech cocycle. It defines the relative line

$$L_{\text{rel}} = \det E_2 \cong (\det E_3)^*.$$

No separate phase-atlas hypothesis is needed.

No-go theorem 11.5 (Curvature-only determinant argument). *The local condition $\text{Tr } F_3 + \text{Tr } F_2 = 0$ does not eliminate a nontrivial flat total determinant line. Controlling all periods requires a global trivial-holonomy condition, supplied here by a parallel unit tensor such as τ_{src} .*

12 Composition of completed regions and Gauss constraints

A passive gauge-link law does not by itself restrict the state space to gauge singlets. To obtain Gauss constraints we specify a *composition rule*: when regions are joined at an internal record, their boundary amplitudes are sewn independently of an arbitrary internal frame. The precise rule uses coherent averaging, and differs from discarding a classical frame label in a density matrix.

Axiom 12.1 (Invariant internal sewing). *At each internal record v , let R_v be the unitary representation of the retained compact connected frame group G on all incident boundary and matter indices. Sewing replaces a boundary amplitude by its normalized Haar average $P_v\psi = \int_G R_v(g)\psi \, dg$. The initial physical sector has no external charge at such an internal record. Boundary records may carry specified charge representations.*

This is additional physical content. Independence of the name of a frame motivates it but does not uniquely force coherent averaging. Different charge sectors or reference-carrying boundaries are legitimate alternatives.

Theorem 12.2 (Invariant sewing gives the Gauss sector). *P_v is the orthogonal projector onto G -invariant amplitudes. If $R_v(e^{sT_a}) = e^{is\mathcal{G}_v^a}$, then*

$$P_v\psi = \psi \iff \mathcal{G}_v^a\psi = 0 \text{ for every generator } a,$$

on the common generator domain. Sewing at different vertices gives the intersection of these Gauss sectors. A Hamiltonian commuting with all $R_v(g)$ preserves the sewn subspace.

Proof. Haar invariance gives $R_v(h)P_v = P_v$, $P_v^* = P_v$, and $P_v^2 = P_v$. Every invariant vector is fixed by the integral, proving the range statement. Differentiating group invariance gives the generator conditions. Conversely zero generators imply invariance under their one-parameter subgroups, which generate a connected group. On a link, the actions of its two endpoints are left and right translations and commute, so the vertex projectors commute. Commutation of a Hamiltonian with the representation implies commutation with each projector. $\square$

On $L^2(G^\mathcal{E})$ the link $U_{wv} : E_v \rightarrow E_w$ transforms actively as $U_{wv} \mapsto h_w U_{wv} h_v^{-1}$. The generator at a vertex is the sum of its signed left/right electric generators plus its matter charge. This is the Hamiltonian Gauss law of lattice gauge theory [12]. Its sign is fixed by this endpoint convention. For $G = S(U(3) \times U(2))$ there are twelve generators per vertex; the two block centers share one relative generator. No thirteenth independent Abelian constraint is introduced.

Proposition 12.3 (Frame twirling is not singlet projection). *The invariant density matrices of $\mathcal{T}(\rho) = \int_G R(g)\rho R(g)^* \, dg$ need not be supported on $\text{Ran } P$. Thus ordinary ignorance of a frame does not establish the singlet Gauss sector.*

Proof. On a nontrivial irreducible representation of dimension d , the density matrix I_d/d is invariant under twirling, while the Haar average of amplitudes is zero because the representation contains no invariant vector. More generally, for $\mathcal{K} = \bigoplus_\lambda V_\lambda \otimes M_\lambda$, twirling produces blocks of the form $I_{V_\lambda}/\dim V_\lambda \otimes \rho_\lambda$; it does not delete nontrivial λ sectors. $\square$

This distinction identifies exactly what must be derived from a future microscopic completion rule. If it supplies coherent internal sewing and the neutral initial sector, Gauss constraints follow. If it supplies only twirling, one obtains invariant density operators with possible charges. The later rotor and singlet-loop models respect the constraint once that sector is selected; symmetry of their Hamiltonians does not independently choose the sector.

13 Local temporal frames and the necessity of a connection

Axiom 13.1 (Smooth realization of operational comparisons). *Use R1–R4 from Section 6 on a smooth record chart. The retained spaces form smooth Hermitian bundles, and their relational comparators preserve the declared tensors. No differential comparison operator is included among these assumptions.*

Corollary 13.2 (Differential comparison law). *The operational necessity theorem gives, in a local frame u with $\Psi = u\psi$,*

$$D\psi = d\psi + A\psi, \quad (48)$$

$$A' = g^{-1}Ag + g^{-1}dg, \quad (49)$$

$$D'\psi' = g^{-1}D\psi, \quad \psi' = g^{-1}\psi. \quad (50)$$

Proof. Apply the first-order expansion of the faithful links in Theorem 6.6. The displayed law follows from changing endpoint frames; it is not inferred merely by defining a Leibniz operator to be a connection. $\square$

The result establishes necessary comparison structure, not nonzero curvature or a propagating force. A gauge-invariant formulation can use relational or loop data directly. The conventional redundancy interpretation is reviewed in [15].

13.1 Protected spectral transport

Let $H_{\text{in}}(x)$ be a smooth finite-dimensional self-adjoint family, or a suitably common-domain operator family with a smooth isolated spectral projector $P(x)$ of constant rank. Then $E = \text{Ran } P$ is a Hermitian bundle. The adiabatic transport mechanism of [14] gives its natural projected connection.

Proposition 13.3 (Intrinsic Kato connection and curvature). *For a section $s = Ps$, define $\nabla s = Pds$. In an orthonormal frame u of E ,*

$$A = u^\dagger du, \quad F = dA + A \wedge A, \quad \mathcal{F} = P(dP \wedge dP)P.$$

Here A is anti-Hermitian and $\mathcal{F} = uFu^\dagger$ on E .

Proof. $u^\dagger u = I$ implies $A^\dagger = -A$. Expanding $\nabla^2 s = Pd(Pds)$ and using $PdPP = 0$ gives $P(dP \wedge dP)Ps$. Local expression yields F . $\square$

The identity $PdPP = 0$ does not imply $\nabla = 0$: differentiation acts on sections, not just on the projector. A fixed projector with a varying frame has a pure-gauge connection and zero curvature. On a one-dimensional base every curvature two-form vanishes; local curvature physics requires a higher-dimensional record or parameter base, while loops may still support global holonomy.

13.2 Compatibility with the determinant tensor

The raw projected connection need not make τ_{src} parallel. This is an extra compatibility condition. In local frames satisfying $\tau_{\text{src}} = 1$, let A_3, A_2 be block-unitary connections and $a = \text{Tr } A_3 + \text{Tr } A_2$. The corrections

$$A_3^{\text{src}} = A_3 - \frac{a}{5}I_3, \quad A_2^{\text{src}} = A_2 - \frac{a}{5}I_2$$

give total trace zero and patch covariantly under $S(U(3) \times U(2))$ transitions. They define one compatible connection; they do not show that microscopic adiabatic transport implements this correction. The raw Kato connection, a source-compatible connection, and an independently variable gauge field must therefore be distinguished.

Proposition 13.4 (Space of compatible connections). *On the reduced principal bundle $\mathcal{Q} \rightarrow X$, source-compatible connections form an affine space modeled on $\Omega^1(X, \text{ad } \mathcal{Q})$. On a trivial bundle, after choosing a reference trivialization, they are represented by $\Omega^1(X, \mathfrak{g}_{\text{src}})$, where*

$$\mathfrak{g}_{\text{src}} = \{(X_3, X_2) : \text{Tr } X_3 + \text{Tr } X_2 = 0\} \cong \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1).$$

The algebra has real dimension $8 + 3 + 1 = 12$.

Proof. The difference of two connections is a global adjoint-valued one-form. Differentiating the unitary stabilizer condition gives the trace constraint and the stated decomposition. $\square$

The source-preserving completion-control theorem below establishes pointwise curvature accessibility in its declared control class. It does not prove that every spacetime connection field is physically realized by the bare source.

13.3 Curvature, holonomy, and relational observables

Parallel transport along γ is $U_\gamma = \mathcal{P} \exp(-\int_\gamma A)$. For a closed loop, its matrix changes by conjugation at the base point; traces and eigenvalues are frame independent. For a small oriented loop, $U_\gamma = I - F_{\mu\nu} \delta S^{\mu\nu} + \dots$ with the area convention fixed by the loop orientation. Curvature measures path-dependent comparison of temporal carriers. Non-Abelian behavior requires noncommuting transport, not merely rank greater than one.

A connection need not realize its full structure group as holonomy. On a base of dimension at least two, a local existence construction can place twelve smooth disjoint curvature patches along one coordinate plane, one for each Lie-algebra basis vector T_a . In radial gauge around each patch center, $A = \frac{1}{2} T_a (x dy - y dx)$ locally gives $F(0) = T_a dx \wedge dy$. Choosing paths outside other patches transports these generators to one base point. Their span supplies full restricted holonomy. This establishes existence of suitable connections, not saturation by the source-induced one.

14 Explicit curved carriers with fixed spectral protection

The finite clock alone has no spatially varying projector. To exhibit nontrivial transport, we now specify additional carrier-embedding data over a smooth chart. The construction is an existence model with controllable responses, not a claim that the clock forces this embedding.

14.1 Graph-projector realization

Let $Z(x)$ be a smooth $m \times m$ complex matrix and put

$$u_Z = \begin{pmatrix} I_m \\ Z \end{pmatrix} (I_m + Z^\dagger Z)^{-1/2}, \quad P_Z = u_Z u_Z^\dagger, \quad H_Z = \delta(I_{2m} - P_Z), \quad \delta > 0. \quad (51)$$

The rank- m zero-energy cluster is isolated by δ everywhere, independently of Z . A realification gives a real orthogonal model with a commuting standard complex structure. All complex scalars can equivalently be regarded as elements of the selected $\mathbb{C}_J$.

Proposition 14.1 (Arbitrary matrix curvature at a graph origin). *If $Z(0) = 0$, then*

$$F(0) = dZ^\dagger \wedge dZ.$$

For $Z(x, y) = xI_m + yB$, with $B^\dagger = -B$, one has

$$F_{xy}(0) = 2B.$$

Thus every prescribed $T \in \mathfrak{u}(m)$ occurs as a local curvature coefficient by setting $B = T/2$.

Proof. At the origin the derivative of the normalizing factor is zero, and $du = (0, dZ)^T$. The projected curvature is the antisymmetric part of $du^\dagger du$, giving $dZ^\dagger \wedge dZ$. Its xy coefficient is $I^\dagger B - B^\dagger I = 2B$. $\square$

For a pair $T = (T_3, T_2) \in \mathfrak{g}_{\text{src}}$ choose a graph for each block. At the origin total trace curvature vanishes. Away from that point the raw graph connections need not have total trace zero; the determinant-compatible correction in Section 13 produces a reduced connection. Since the correction is central and the total trace of the origin curvature is zero, the prescribed origin curvature is unchanged. This gives an explicit local witness for all twelve allowed Lie-algebra directions. It is stronger than asserting that some curvature may exist, but it does not select a unique physical source family.

14.2 Clock spectrum and carrier transport are distinct data

The graph Hamiltonian has a degenerate protected band, whereas the clock has unequal Fourier frequencies. They should not be identified. One may embed the clock triplet isometrically as $u_3 \text{diag}(\omega_1, \omega_2, \omega_3) u_3^\dagger$ with a sufficiently separated complementary band, preserving the clock eigenvalues. Its projector geometry remains the same, but autonomous adiabatic transport includes internal dynamical phases and possible splitting-dependent effects. The polar-overlap comparison protocol below measures the cluster geometry directly. Interpreting it as physical adiabatic transport requires either a degenerate carrier Hamiltonian or control of internal evolution, such as an explicitly specified echo. A record map that forgets selectors does not by itself erase a right Hamiltonian's spectral splitting.

15 Which gauge directions can the temporal source access?

An allowed frame group is not a controllability theorem. This section distinguishes variation of the bare clock frame from variation of a retained open-completion embedding. The distinction prevents a full Lie algebra from being inferred solely from its dimension.

Proposition 15.1 (Bare source-frame variation is geometrically flat). *In the explicit clock of Definition 8.4, let only the oriented source frame $F(x) \in SO(3)$ vary, with the cosine/sine copies and fixed line identified across the chart. Then the positive-frequency rank-three projector is constant. The doubled real defect projector has a global real orthonormal basis with zero projected connection. Its scalar extension through the constant selected phase algebra is also flat. Thus source-frame variation alone does not generate the curved graph carriers of Section 14.*

Proof. The frame change acts as $\text{diag}(1, F, F)$ on $\mathbb{R}u_0 \oplus W_c \oplus W_s$. All three oriented rotation angles lie in $(0, \pi)$, so the polar phase acts by the same quarter-turn on each cosine/sine pair and commutes with $\text{diag}(F, F)$. The positive-frequency projector depends only on that phase and is therefore independent of F . For the defect use normalized real vectors $d_-(x) = (K_x^{-1}\Omega_x, 0)$ and $d_+(x) = (0, K_x\Omega_x)$ in the two doubled summands. Their diagonal inner products with their derivatives vanish because they are real unit vectors; cross inner products vanish because the summands are orthogonal. Hence their projected connection is zero, as is the connection after scalar extension. $\square$

This result concerns the stated restricted variation. General source conjugacies that mix fixed and dynamic sectors can change the triplet projector, and additional completion embeddings can change both retained carriers. Neither type of variation follows merely from choosing another oriented quaternionic source frame.

15.1 Source-preserving open-completion controls

Define an admissible completion-control class as follows:

- (A1) The intrinsic marked source clock remains the seven-dimensional regular pair (K_0, Ω_0) , up to an isometric transport of its entire real source space. The no-spectator rule applies to this intrinsic clock, not to every auxiliary record environment.

- (A2) The selected triplet and scalar-extended doublet are embedded independently by smooth complex-linear isometries $u_3 : \mathbb{C}_J^3 \rightarrow \mathbb{C}_J^6$ and $u_2 : \mathbb{C}_J^2 \rightarrow \mathbb{C}_J^4$. The extra dimensions are declared open-completion modes. Their physical existence is an added hypothesis.
- (A3) The intrinsic source operators, real cut, record channel, and total-volume reference are transferred consistently under those isometries. The source-compatible comparator includes the declared determinant correction when needed.
- (A4) At a chart origin the allowed embedding variations include $Z_m = xI_m + yB_m$ for arbitrary anti-Hermitian B_m , subject to the paired source curvature direction $2 \operatorname{Tr} B_3 + 2 \operatorname{Tr} B_2 = 0$.

A4 is a controllability hypothesis about the completion environment. It is stronger than the original source axioms and can be tested independently of the source spectrum.

For clarity, transport of the intrinsic source clock can be made explicit. Regard its nonfixed real sector as the underlying real space of $\mathbb{C}_J^3$. Let ι_x map its fixed line to a constant unit vector and its nonfixed sector through the realification of $u_3(x)$. Then

$$\iota_x^* \iota_x = I_7, \quad K_x|_{\operatorname{Ran} \iota_x} = \iota_x K_0 \iota_x^*, \quad \Omega_x = \iota_x \Omega_0.$$

The marked orbit within this invariant source subspace remains regular with first return seven. Complementary environmental modes do not enter its orbit. The real cut defects are obtained inside that transported source subspace, extended through $\mathbb{C}_J$, and then mapped into the doublet completion carrier by u_2 . This is an explicit interface isomorphism rather than identification of doublet states with vectors inside the triplet.

The graph band Hamiltonians can maintain a fixed protective gap independently of the clock spectrum. More specifically, for a fixed Hermitian carrier matrix h_m with spectrum separated from δ ,

$$\tilde{H}_m(x) = u_m(x) h_m u_m(x)^\dagger + \delta[I - P_m(x)]$$

has exactly the same eigenvalues and cluster separation at every point. The connection is associated with the whole cluster; interpreting it as autonomous adiabatic evolution of a nondegenerate cluster still requires control of internal dynamics.

Theorem 15.2 (Twelve-direction local curvature accessibility). *In A1–A4, every $T = (T_3, T_2) \in \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(2))$ is the origin curvature $F_{xy}(0)$ of a source-preserving completion family with unchanged intrinsic seven-cycle and unchanged carrier eigenvalues. The control-to-curvature map $B \mapsto 2B$ is onto the twelve-dimensional algebra. In orthonormal Lie-algebra coordinates, setting $B = T/2$ makes this map the identity.*

Proof. Choose $Z_m = xI_m + yT_m/2$. Proposition 14.1 gives origin curvature T_m . Since $\operatorname{Tr} T_3 + \operatorname{Tr} T_2 = 0$, the total-trace correction does not change that origin curvature. The isometric source transport preserves the marked orbit and cut algebra, and the displayed cluster Hamiltonian preserves eigenvalues and separation. The map is linear with full rank twelve. $\square$

This proves pointwise accessibility in a declared completion-control class. It does not prove that arbitrary spacetime connection fields, global topology, or a quantum measure are generated by the bare source. Universal connection realization by Stiefel-bundle pullback is already a classical result [5]; the present calculation is a finite constrained witness preserving the specific source and defect interfaces.

Corollary 15.3 (Stable accessibility under small control-map errors). *Let $L_0 = I_{12}$ be the normalized origin control-to-curvature map and let a perturbed real-linear map obey $\|L - L_0\|_{\operatorname{op}} \leq \varepsilon < 1$. Then it remains onto, $\sigma_{\min}(L) \geq 1 - \varepsilon$, and a prescribed curvature direction can be reached with control norm at most $\|T\|/(1 - \varepsilon)$.*

Proof. $\|Lv\| \geq \|v\| - \|(L - I)v\| \geq (1 - \varepsilon)\|v\|$. In finite equal dimensions this implies invertibility and the stated inverse-norm bound. $\square$

Small perturbations must also preserve cluster isolation and the retained tensor conditions. The estimate controls accessibility, not physical renormalization or gauge-field propagation. A measured rank-deficient control map would reject A4 without changing the exact intrinsic source recurrence.

16 Inner-time recurrence and the spatial-dimension gap

The SMM/DTT interpretation seeks to obtain spatial multiplicity from inner generation and completed records. The seven-state recurrence here selects three positive-frequency modes and one reset state, but matching these counts to three spatial directions and an outer time does not provide a canonical displacement map: the reset state is not derived as a macroscopic time coordinate, and a spectral mode is not a direction of motion between records. The quaternionic source has a three-dimensional real imaginary module, and its reflection-odd clock output is a copy of that module. Neither statement identifies it with spatial displacements between completed records. The following independent counterexample makes the distinction precise.

Proposition 16.1 (Spatial dimension is independent of the finite clock). *For each integer $q \geq 1$, the seven-state source, scalar-extended defect, standard-form record channel, and retained product-volume tensor admit a translation-invariant realization over a record graph with spatial volume growth of degree q and random-walk return exponent $q/2$. The comparisons may be flat in every realization. Consequently the finite source and declared local record operations do not select three spatial dimensions, even when the outer records are assigned a common chronological index.*

Proof. Take $G_q = (\mathbb{Z}^q, E_q)$ with its nearest-neighbor edges. At each vertex use an identical copy of the finite source, defect, and local channel from Sections 8 to 10. Equip each retained carrier with the same total-volume tensor, and choose identity comparators along edges in a product trivialization. These are tensor compatible and have trivial holonomy; a change of independently chosen vertex frames gives the covariant links $g_x^{-1}g_y$. No vertex records a distinguished internal basis. The number of vertices within graph distance r is $|B_x(r)| = \Theta(r^q)$. For nearest-neighbor simple random walk, Fourier inversion gives

$$p_{2n}(x, x) = \frac{1}{(2\pi)^q} \int_{[-\pi, \pi]^q} \left(\frac{1}{q} \sum_{i=1}^q \cos k_i \right)^{2n} d^q k = \Theta(n^{-q/2}).$$

The exponents vary with q , while all finite source and channel identities remain the same. Copies indexed by an additional discrete outer time n preserve the same conclusion on each chronological slice; the index alone specifies neither a Lorentzian metric nor a causal light cone. $\square$

The counterexample does not say that q -dimensional graphs are generated by DTT. It shows that the present axioms leave their adjacency unspecified. If a microscopic update is proposed to form adjacency rather than assuming it, its dimension claim should pass the following independent tests:

- (D1) **Derived record graph.** Define completed events, a source-induced relation of causal influence or finite-range comparison, and its adjacency and edge weights using the microscopic update alone. Prove that the rule selects a robust large-scale graph class without stipulating a cubic lattice.

- (D2) **Spatial scaling.** For growing finite graphs of linear scale L and $\Theta(L^3)$ vertices, prove $|B_x(r)| \asymp r^3$ on expanding ranges $1 \ll r \ll L$, and for a specified unbiased diffusion prove a corresponding return law $p_{2n}(x, x) \asymp n^{-3/2}$ for $1 \ll n \ll L^2$, with uniform constants away from boundaries. The second exponent is the spatial spectral dimension diagnostic, not the dimension of the seven-state Hilbert space [13]. Neither scaling law by itself establishes a smooth three-manifold.
- (D3) **Source-to-displacement map.** Construct from the same update an operational displacement or coframe map into the reflection-odd source module $W_s \simeq \text{Im } \mathbb{H}$. Prove that its continuum limit has rank three, a nondegenerate metric induced by the quaternionic norm, and transition functions that identify source $SO(3)$ with rotations of *spatial tangent frames*. Without this map, the $SO(3)$ of Lemma 9.1 remains an internal output isotropy. Verify in the joint record channel that the spatial source-frame action and the retained internal right-frame action remain independent: the coframe must not select a preferred mixed spatial-internal tensor that reduces the claimed gauge stabilizer.
- (D4) **Causal limit.** Derive chronological ordering, an appropriate local causal propagation rule, and a four-dimensional Lorentzian continuum with three-dimensional spatial slices. Establish this limit in addition to the two spatial scaling tests; a discrete clock or a separately attached time index does not imply a causal metric.

These are proof obligations, not results of the present finite construction. They also keep distinct the split-complex temporal symbol j , when used in DTT with $j^2 = +1$, and the selected ordinary complex phase J with $J^2 = -I$ in Section 9. The powers of j alternate between 1 and j ; without an additional dynamical representation they do not generate three spatial displacement directions. A bridge between these two structures must specify its action on completed-event correlations and establish the spatial and causal limits above.

16.1 An operational displacement criterion

To turn the source module into space, attach to a comparison history a displacement $\xi(h) \in W = \text{Im } \mathbb{H}$. In a flat spatial presentation require

$$\xi(h_2 h_1) = \xi(h_2) + \xi(h_1), \quad \xi(h^{-1}) = -\xi(h). \quad (52)$$

This is a candidate dynamical output, not a definition of spatial dimension. If it vanishes on closed spatial comparison histories, it integrates to $f(v) = \xi(h_{o \rightarrow v})$, independently of the path. Nonzero translational periods require local charts or a covering space. With varying spatial frames, the addition law uses the appropriate spatial parallel transport; the flat formula does not impose zero gauge curvature on the independent internal carriers.

Proposition 16.2 (A sufficient three-dimensional growth criterion). *Suppose a connected bounded-degree record graph admits a map $f : \mathcal{V} \rightarrow W$ with uniformly bounded fibers, whose distinct image points are uniformly separated and form a relatively dense subset of W . Assume constants $a \geq 1, b \geq 0$ satisfy*

$$a^{-1} d_\Gamma(v, w) - b \leq |f(v) - f(w)| \leq a d_\Gamma(v, w) + b.$$

Then its graph balls obey $|B_v(r)| \asymp r^3$ for large r .

Proof. The right inequality places the image of a graph ball in a Euclidean ball of radius $ar + b$. Uniform separation bounds the number of image points there by a constant times r^3 , and bounded fibers give the same upper bound for vertices. Relative density supplies a constant times r^3 image points in a Euclidean ball of suitably smaller radius around $f(v)$. The left inequality puts preimages of those points inside the graph ball after decreasing that radius by a fixed factor and additive constant. This proves the lower bound. $\square$

The proposition explains why a source-derived map would matter: an operationally faithful, noncollapsed displacement geometry transfers the source's dimension to record growth. Merely writing a map into W is insufficient. Relative density, bounded fibers, and the two-sided distance control are substantial dynamical requirements. They exclude hidden macroscopic directions and collapsed or space-filling encodings.

Diffusion is a second test. In the explicit translation-invariant case, let a symmetric finite-range lazy walk on $\mathbb{Z}^3$ have step probabilities $p(s)$, with $p(0) > 0$, generating support, and positive covariance $\Sigma = \sum_s p(s) s s^T$. Its characteristic function is

$$\hat{p}(k) = 1 - \frac{1}{2} k^T \Sigma k + O(|k|^4).$$

Fourier inversion, exponential suppression away from $k = 0$, and the change $k = n^{-1/2} q$ give

$$p_n(0, 0) \sim \frac{1}{(2\pi n)^{3/2} \sqrt{\det \Sigma}}. \quad (53)$$

This is a self-contained diffusion witness, not a claim that temporal generation has already produced its lattice or step distribution. On irregular networks one must prove appropriate heat-kernel control; volume growth alone does not rule out bottlenecks and anomalous diffusion.

An additional smooth limit can turn ξ into a coframe $e : T\Sigma \rightarrow W$ of rank three. Then $h(X, Y) = \langle e(X), e(Y) \rangle_W$ is a spatial metric. The source $SO(3)$ acts on this coframe, while the independent right $S(U(3) \times U(2))$ action acts on internal fibers. Deriving this factorization is indispensable: equating spatial rotations with the weak gauge group would change the physical symmetry rather than explain it.

16.2 Completion delays give a cone after displacement is defined

Suppose a completed history has displacement bounded by a length budget, $|\xi(h)| \leq \ell_* w(h)$, and completion delay $\Theta(h)$ satisfying $\ell_* w(h) \leq c_* \Theta(h)$. Then

$$|\xi(h)| \leq c_* \Theta(h). \quad (54)$$

The proof is the composition of the two inequalities. This is the completion-cone construction of [7], restated after its displacement input. Finite-range many-body propagation bounds provide possible microscopic precedents [27, 28], but an exponential Lieb–Robinson tail is not an exact relativistic light cone and its velocity is not automatically the speed of light.

If a smooth nonvanishing clock one-form ϑ and rank-three spatial coframe exist, then $g = -c_*^2 \vartheta^2 + \sum_{i=1}^3 (e^i)^2$ is Lorentzian when these four forms are independent. This is a conditional representative of the cone. It does not establish that all matter species share its limiting speed, that boosts are dynamical symmetries, or that the metric satisfies a gravitational field equation. Those are separate tests of a physical continuum limit.

16.3 A dimension constraint from electric and magnetic completion

A complementary route constrains dimension from the kinds of sources that a gauge field can support. It uses the degree counting of ordinary differential-form gauge theory [29]. The conclusion is conditional on a continuum description and on the actual existence of the stated defects.

Theorem 16.3 (Point electric and magnetic sources select four spacetime dimensions). *In an oriented spacetime of dimension D , let A be a p -form gauge potential with field strength of degree $p + 1$. An electrically charged object has spatial dimension $p - 1$, while an ordinary localized magnetic source has spatial dimension $D - p - 3$. If both are point particles for the same one-form gauge sector, then $D = 4$. Equivalently, electric and magnetic potentials can both be one-forms only in $D = 4$.*

Proof. Electric coupling integrates A over a p -dimensional worldvolume, so its spatial dimension is $p - 1$. Magnetic charge is the flux of F through a linking $(p + 1)$ -sphere. A defect of spatial dimension d_m in a $(D - 1)$ -dimensional spatial slice has linking sphere dimension $D - d_m - 2$, hence $d_m = D - p - 3$. Pointlike electric charge gives $p = 1$; pointlike magnetic charge then gives $0 = D - 4$. Alternatively, the dual field strength $*F$ has degree $D - p - 1$, so its local potential has degree $D - p - 2$; setting both potential degrees to one again gives $D = 4$. $\square$

The degree-counting theorem is standard, not a new property of DTT. The proposed foundational use is more specific: if a pre-geometric completion rule produces both electric and magnetic defects by the same point-record mechanism, and its continuum field is an ordinary one-form, the dimension is constrained without first stipulating three spatial axes. To make that use noncircular, one must derive both point-source properties from the update rule. A single pre-spatial source does not by itself imply a pointlike magnetic defect, and gauge theory does not in general require dynamical monopole particles. Higher-form gauge fields, extended defects, and sectors without magnetic particles remain alternatives.

This constraint and Proposition 16.2 answer different questions. The first restricts a continuum dimension if a particular source duality exists. The second tests whether completed records actually have the required macroscopic geometry. A credible dimensional reconstruction should make them agree using the same microscopic rule, rather than substituting either for the other.

16.4 Compact phase, topology, and the limits of a single sphere

For a continuous compact $U(1)$ line bundle with connection, magnetic flux through a two-sphere is integral in the convention $\frac{1}{2\pi} \int_{S^2} f \in \mathbb{Z}$. On two hemispheres, Stokes' theorem reduces this integral to the winding of the equatorial transition function. A fixed finite group of phases has only locally constant transition functions; over S^2 it cannot supply a nonzero first Chern class. The seven-state source can select a continuous complex scalar structure, but a nontrivial continuous phase atlas remains global data, not a consequence of the seven labels alone.

Nor can every Abelian field be represented by a single endpoint map $n : X \rightarrow S^2$. If $f = n^*\omega$ for a two-form ω on the sphere, then

$$f \wedge f = n^*(\omega \wedge \omega) = 0.$$

General four-dimensional Abelian fields need not obey this condition. Likewise the square of the pulled-back first Chern class vanishes. A single Hopf-sphere endpoint construction is therefore a restricted carrier class, even though histories of sphere paths can realize many holonomies. Larger carrier spaces or genuinely retained transport data can remove the pointwise restriction; their global topology must still be checked.

17 Finite records, overlap links, and comparison curvature

Let u_x and u_y be orthonormal frames for nearby rank- m retained spaces. Define the overlap and its unitary polar factor by

$$M_{xy} = u_x^\dagger u_y, \quad W_{xy} = M_{xy}(M_{xy}^\dagger M_{xy})^{-1/2}. \quad (55)$$

The link is defined whenever M_{xy} is invertible. For sufficiently close projectors this follows from the principal angles being less than $\pi/2$. Singular overlaps are outside the admissible comparison regime.

Proposition 17.1 (Optimal metric comparison selects the polar link). *For an invertible overlap M_{xy} , its unitary polar factor is the unique minimizer of $\|M_{xy} - V\|_{\text{HS}}$ over $V \in U(m)$. Equivalently it minimizes $\|u_y - u_x V\|_{\text{HS}}$.*

Proof. Both squared objectives depend on V only through $-2 \operatorname{Re} \operatorname{Tr}(V^\dagger M)$. Write $M = L \Sigma R^\dagger$. For unitary V , put $Q = L^\dagger V R$. Then $\operatorname{Re} \operatorname{Tr}(V^\dagger M) = \sum_j \sigma_j \operatorname{Re} \overline{Q_{jj}} \leq \sum_j \sigma_j$. Since all $\sigma_j > 0$, equality forces every $Q_{jj} = 1$, and unitarity forces $Q = I$. Thus $V = L R^\dagger$, the polar factor. $\square$

The metric-optimal polar factor is established matrix analysis [6]. Once a common ambient inner-product geometry and nearest-unitary comparison are selected, the retained overlaps determine a unique discrete comparator. Its infinitesimal limit yields the projected connection. The selection of this metric comparison protocol is an assumption, but its connection law and local coefficients are outputs. A physical derivation must show why this protocol describes completed records.

Theorem 17.2 (Exact discrete gauge covariance). *Under independent frame changes $u_x \mapsto u_x g_x$, $u_y \mapsto u_y g_y$,*

$$W_{xy} \mapsto g_x^\dagger W_{xy} g_y.$$

For a closed polygon $\mathcal{C} = (x_0, \dots, x_N = x_0)$ the product

$$W_{\mathcal{C}} = W_{x_0 x_1} \cdots W_{x_{N-1} x_0}$$

changes only by conjugation at x_0 . Its traces, spectrum, and real trace deficit are frame independent. Along a smooth short displacement,

$$W_{x, x+dx} = I + A_\mu dx^\mu + O(|dx|^2).$$

Proof. $M'_{xy} = g_x^\dagger M_{xy} g_y$ and functional calculus gives $(M'^\dagger M')^{-1/2} = g_y^\dagger (M^\dagger M)^{-1/2} g_y$. Intermediate frames cancel in a closed product. Since $u_x^\dagger \partial_\mu u_x = A_\mu$ is anti-Hermitian, the first-order overlap is already unitary to that order. $\square$

These products compare the endpoint frames back to the starting frame. Their continuum orientation is inverse to forward state transport $\mathcal{P}e^{-\int A}$; the curvature sign reverses under loop reversal, while the trace cost below does not.

17.1 Determinant-compatible discrete links

Raw block polar factors need not satisfy $\det W_3 \det W_2 = 1$. On a small-link chart choose $\theta_{xy} = \operatorname{Arg}(\det W_3 \det W_2)$ continuously near zero and set

$$\widehat{W}_3 = e^{-i\theta_{xy}/5} W_3, \quad \widehat{W}_2 = e^{-i\theta_{xy}/5} W_2.$$

Their total determinant is one. Under source-adapted frame changes, the total determinant and its selected small-link branch are invariant, so the links retain exact covariance. The continuum correction is precisely subtraction of the total trace divided by five. Large links and nontrivial global bundles require compatible patching; a fifth-root branch is not automatically global. This explicit correction is an imposed source-compatible comparison rule, not a property of every uncorrected spectral overlap.

17.2 A primitive single-carrier loop test generates the comparison cost

The loop cost can be grounded in a specified comparison experiment before assigning it energy. Prepare a carrier in an unpolarized state $\rho_m = I_m/m$ and a two-path auxiliary probe in $|+\rangle$. Apply W_p to the carrier in one path and identity in the other, then measure the probe in the $|\pm\rangle$ basis. The probe supplies a declared comparison reference; it is not the phase register whose selector coherences are erased.

Proposition 17.3 (Isotropic single-carrier comparison rejection). *The rejection probability is*

$$p_{-,m}(W_p) = \frac{1}{2} \left[1 - \frac{1}{m} \operatorname{Re} \operatorname{Tr} W_p \right].$$

The maximally mixed input is the unique density matrix invariant under all adapted block frames. If independent loop comparisons in block m incur penalty $2m\kappa_m$ per rejection, their mean penalty is exactly $\kappa_m[m - \operatorname{Re} \operatorname{Tr} W_p]$.

Proof. The off-diagonal probe density-matrix entry is $\operatorname{Tr}(\rho_m W_p)/2$ up to complex conjugation, so the minus-port probability has the displayed form. By irreducibility of each $SU(m)$ defining representation, a density matrix invariant under all its frame transformations is scalar; unit trace fixes I_m/m . Multiplication by the stated rejection penalty gives the cost. $\square$

Thus the trace functional is an output of one explicit isotropic single-carrier interference protocol. The existence of that protocol, its reference, independent additive penalties, and any energy interpretation are assumptions. Collective determinant probes, multi-carrier tests, or nonlinear classical processing can generate additional invariant costs. The result neither assumes a gauge-field kinetic action nor derives the cost coefficient from temporal unity.

17.3 An explicit comparison cost and its curvature limit

For a four-dimensional Euclidean cubic record lattice of spacing a , use source-compatible links and define the elementary plaquette product $\widehat{W}_{p,m}$ in each block. Choose the local comparison functional

$$\mathcal{E}_a = \sum_p \sum_{m \in \{3,2\}} \kappa_m \left[m - \operatorname{Re} \operatorname{Tr} \widehat{W}_{p,m} \right], \quad \kappa_m > 0. \quad (56)$$

For a unitary loop comparator,

$$m - \operatorname{Re} \operatorname{Tr} W_p = \frac{1}{2} \|I_m - W_p\|_{\text{HS}}^2.$$

Thus the functional sums squared metric discrepancies between identity comparison and closed-loop comparison. This is the mean rejection penalty of the primitive protocol. Assigning those penalties physical energy, and selecting their coefficients, are additional microscopic assumptions. It is a Wilson-type functional [11]. Introducing it does not constitute an independent derivation of the Wilson action from DTT alone.

Proposition 17.4 (Positive curvature cost from polar-overlap comparison). *For smooth source-compatible carriers and uniformly admissible short overlaps,*

$$m - \operatorname{Re} \operatorname{Tr} \widehat{W}_{p,m} = \frac{a^4}{2} \|F_{\mu\nu,m}\|_{\text{HS}}^2 + O(a^5).$$

On a fixed smooth four-dimensional region the leading continuum functional is

$$\mathcal{E}_0 = \frac{1}{2} \int d^4x \sum_{\mu < \nu} [\kappa_3 \|F_{\mu\nu,3}\|_{\text{HS}}^2 + \kappa_2 \|F_{\mu\nu,2}\|_{\text{HS}}^2]. \quad (57)$$

Proof. The logarithm of the unitary small plaquette is $a^2 F_{\mu\nu} + O(a^3)$, anti-Hermitian. The real trace of its linear term vanishes; its quadratic term is $a^4 \operatorname{Tr}(F^2)/2 = -a^4 \|F\|_{\text{HS}}^2/2$. Higher cross terms begin at order a^5 . Multiplication by the number of plaquettes per physical volume, of order a^{-4} , yields the displayed Riemann sum limit. $\square$

The limit is a deterministic smooth-field limit. It does not construct a quantum path-integral measure, establish regulator-uniform correlation bounds, or prove a Yang–Mills continuum theory. Four-dimensionality is assumed here rather than derived from the record graph. A different microscopic cost can produce additional terms or different coefficients.

17.4 A falsifiable coefficient relation within the two-block cost model

Write the central part as $F_3 = F_3^0 - 2ifI_3$, $F_2 = F_2^0 + 3ifI_2$, with traceless anti-Hermitian F_m^0 and real two-form f . Then

$$\|F_3\|^2 = \|F_3^0\|^2 + 12f^2, \quad (58)$$

$$\|F_2\|^2 = \|F_2^0\|^2 + 18f^2. \quad (59)$$

Consequently the pure central stiffness in the chosen normalization is

$$\kappa_1 = 12\kappa_3 + 18\kappa_2. \quad (60)$$

This is a two-parameter model constraint and can be tested by comparing infinitesimal trace-deficit costs in the two traceless sectors and in the central direction. General gauge-invariant effective actions admit an independent central coefficient, for example from a separate determinant-plaquette cost. Hence failure of (60) rejects this particular two-block comparison-cost model, not gauge symmetry or SMM/DTT in general. Renormalization need not preserve the relation. The coefficients are not identified with observed gauge couplings.

17.5 Exact stability class and an independent-cost diagnostic

Let the quadratic response functional for one curvature component be $Q(F)$. In the primitive independent-block class,

$$Q(F) = \kappa_3 \|F_3\|_{\text{HS}}^2 + \kappa_2 \|F_2\|_{\text{HS}}^2, \quad \kappa_m > 0.$$

The central stiffness residual is

$$\mathcal{C} = \kappa_1 - 12\kappa_3 - 18\kappa_2. \quad (61)$$

Theorem 17.5 (Closure of the primitive quadratic response class). *Any correction that changes only the independent block test rates or positive rejection penalties, and any analytic cost addition with zero Hessian at identity, preserves $\mathcal{C} = 0$ at quadratic order. This includes arbitrary independent changes of κ_3 and κ_2 . It does not include every gauge-invariant correction.*

Proof. Changes of block rates or penalties retain the two-trace form. Analytic additions with zero Hessian do not alter its quadratic coefficients. Expanding along the central generator gives the factors twelve and eighteen, so the residual is identically zero throughout this class. $\square$

This is a structural closure result, not a radiative Ward identity. In an arbitrary quantum field theory, fluctuations may leave this cost class.

Proposition 17.6 (The missing invariant and its measurable coefficient). *For a constant gauge-invariant positive quadratic form on the declared Lie algebra, the most general response can be written*

$$Q(F) = \kappa_3 \|F_3\|^2 + \kappa_2 \|F_2\|^2 + \chi |\text{Tr } F_3|^2.$$

Here $\kappa_3, \kappa_2 > 0$ and the central coefficient must remain positive. Its residual obeys

$$\mathcal{C} = 36\chi.$$

Thus a nonzero residual measures an additional central comparison cost that the primitive independent-block protocol does not contain.

Proof. The compact Lie algebra decomposes into two nonisomorphic simple ideals and one central line. An invariant symmetric quadratic form has independent coefficients on those three summands and no cross terms between them. The two full-block traces span a two-dimensional subspace of these forms. On the central direction $F_3 = -2ifI_3$, $F_2 = 3ifI_2$, one has $|\text{Tr } F_3|^2 = 36f^2$, which supplies the third coefficient. $\square$

χ may be negative provided the total central coefficient is positive; it is not by itself a separate positive penalty. Additional invariant terms are not forbidden by gauge symmetry.

Proposition 17.7 (Sharp small-error bound for the stiffness residual). *Use the direct-sum Hilbert–Schmidt norm, and let H_0 be a primitive invariant quadratic-response Hessian. If another invariant Hessian satisfies $\|H - H_0\|_{\text{op}} \leq \eta$, then*

$$|\mathcal{C}| \leq 60\eta.$$

For measured coefficients with absolute calibration errors $\epsilon_1, \epsilon_3, \epsilon_2$, the measured residual differs from the true residual by at most $\epsilon_1 + 12\epsilon_3 + 18\epsilon_2$.

Proof. The central generator $i \text{diag}(-2I_3, 3I_2)$ has squared norm thirty. Hence $|\delta\kappa_1| \leq 30\eta$, whereas each unit-normalized simple-sector coefficient changes by at most η . The triangle inequality gives $30\eta + 12\eta + 18\eta = 60\eta$. Invariant shifts of $+\eta$ on the normalized center and $-\eta$ on both simple ideals attain the bound when baseline positivity margins allow them. Calibration errors give the second bound by the same weighted triangle inequality. $\square$

Here H denotes the operator representing the quadratic form $Q(F) = \langle F, HF \rangle$, not twice that operator under an alternative calculus convention. This specification fixes the factor sixty. The bound is meaningful only with an independently assessed correction budget η and the stated normalization. It does not predict the numerical sizes of observed gauge couplings.

18 Leakage, curvature, and the cost of temporal comparison

The same projector that determines curvature also controls the loss under successive projections. Put

$$B_\mu = (I - P)\partial_\mu u, \quad q_\mu = \|B_\mu\|_{\text{HS}}^2.$$

For a displacement ϵ in direction μ ,

$$\text{Tr}[I - M^\dagger M] = \epsilon^2 q_\mu + O(\epsilon^3). \quad (62)$$

The left side is the total projection loss summed over an orthonormal initial carrier basis, not its normalized average; divide by m for a maximally mixed state’s loss.

Theorem 18.1 (Curvature bounded by cross-direction projection leakage). *For a smooth Hermitian projector bundle,*

$$F_{\mu\nu} = B_\mu^\dagger B_\nu - B_\nu^\dagger B_\mu, \quad \|F_{\mu\nu}\|_{\text{HS}} \leq 2\sqrt{q_\mu q_\nu}. \quad (63)$$

Consequently an uncorrected polar-overlap plaquette satisfies

$$m - \text{Re Tr } W_p \leq 2a^4 q_\mu q_\nu + O(a^5).$$

Proof. The curvature identity follows by inserting $I - P$ between frame derivatives. The Hilbert–Schmidt product estimate gives $\|B_\mu^\dagger B_\nu\|_{\text{HS}} \leq \|B_\mu\|_{\text{HS}} \|B_\nu\|_{\text{HS}}$, and the triangle inequality gives the bound. The plaquette statement follows from its quadratic expansion. $\square$

This inequality tests whether a measured loop signal and measured projection losses can arise from the same smooth retained projector. It is a geometric constraint, not a DTT-exclusive effect. For the determinant-corrected pair the blockwise bound need not persist separately, because curvature is redistributed centrally. In the direct-sum Hilbert–Schmidt norm the correction is orthogonal projection onto the total-trace-zero algebra and cannot increase the pair’s norm; hence the corresponding sum-of-blocks bound survives.

For the graph model at its origin, $B_x = I_m$, $B_y = B$, so $q_x = m$, $q_y = \|B\|^2$ and $F_{xy} = 2B$. The gap δ remains fixed while the curvature varies. Thus a nonzero curvature signal is compatible with spectral protection and does not require gap closure.

19 Calculated finite-record consequences beyond the seven-cycle ratio

Each prediction in this section is conditional on a completely specified projector or record channel. None is advertised as a numerical Standard-Model prediction. The advantage is that failure has a clear mathematical interpretation and need not be absorbed into an unspecified fit.

19.1 Latitude-loop phase and projection survival

Take a rank-one carrier in $\mathbb{C}^2$,

$$u(\vartheta, \varphi) = \begin{pmatrix} \cos(\vartheta/2) \\ e^{i\varphi} \sin(\vartheta/2) \end{pmatrix}, \quad b = \sin^2(\vartheta/2). \quad (64)$$

In the anti-Hermitian frame convention used in this paper, $A = u^\dagger du$ is the connection one-form. Direct differentiation gives

$$A = i b d\varphi, \quad F = dA = \frac{i}{2} \sin \vartheta d\vartheta \wedge d\varphi. \quad (65)$$

Here φ is the coordinate angle; Φ_N below is an integrated discrete loop phase, not a differential one-form. This is a standard Berry-type carrier [10]; the finite-count formulas below specify the chosen discrete record protocol. For N equal steps at fixed latitude, $h = 2\pi/N$ and

$$m_N = 1 - b + be^{ih}.$$

Proposition 19.1 (Exact phase and survival law). *For nonsingular links with a continuous phase branch, the unwrapped polar-link loop phase and successive-projection survival probability are*

$$\Phi_N = N \operatorname{Arg}(1 - b + be^{2\pi i/N}), \quad (66)$$

$$S_N = [1 - 4b(1 - b) \sin^2(\pi/N)]^N. \quad (67)$$

For fixed b and large N ,

$$\Phi_N = 2\pi b - \frac{(2\pi)^3}{6N^2} b(1 - b)(1 - 2b) + O(N^{-4}), \quad (68)$$

$$-\log S_N = \frac{4\pi^2}{N} b(1 - b) + O(N^{-3}). \quad (69)$$

Thus the geometric phase error is generically N^{-2} while the projection loss is N^{-1} .

Proof. Every overlap is the same m_N . Multiplying its unit polar phase yields the first formula, while multiplying the squared moduli of the rank-one projection amplitudes yields the second. The imaginary Taylor series of $\log(1 - b + be^{ih})$ gives the phase expansion. Expanding $\log[1 - 4b(1 - b) \sin^2(\pi/N)]$ gives the survival expansion. $\square$

At $b = 1/2$ the leading phase error vanishes and, for $N > 2$, the unwrapped phase is exactly π . At $b = 0, 1$ survival is one. These degenerate cases should not be used to infer a generic error exponent. Finite overlap singularities are detected by the survival formula itself. Projection survival assumes an ideal sequence of projective comparisons with internal dynamical phases controlled. It is not the probability law of an arbitrary dissipative record process.

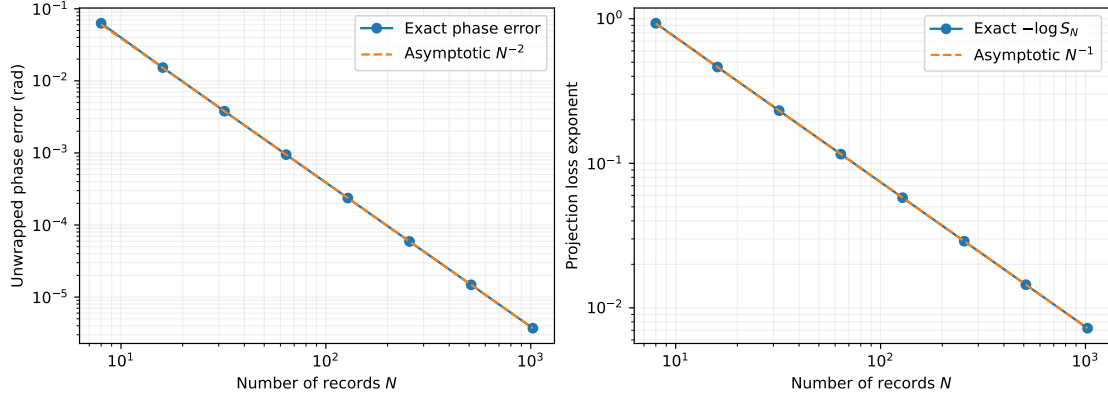


Figure 1: Calculated latitude-record outputs for $b = 1/4$ and $N = 8, 16, 32, 64, 128, 256, 512, 1024$. Left: absolute phase error $|\Phi_N - \pi/2|$, with the leading $(2\pi)^3 b(1-b)|1-2b|/(6N^2)$ guide. Right: $-\log S_N$, with the leading $4\pi^2 b(1-b)/N$ guide. Points use the exact overlap/projection formulas; lines are asymptotic expressions. These are calculated model outputs, not observed data or fits to physical measurements.

19.2 An exactly determinant-linked central carrier family

Let L be the latitude line bundle with connection a . Form retained blocks

$$E_3 = (L^{-2})^{\oplus 3}, \quad E_2 = (L^3)^{\oplus 2}. \quad (70)$$

Here negative powers use the Hermitian dual connection. Then

$$\det E_3 \otimes \det E_2 = L^{-6} \otimes L^6 \cong \mathbb{C},$$

with a canonical parallel pairing tensor. This provides an explicit global realization of the total determinant condition for this central family. It does not prove global descent for an arbitrary seven-cycle source family. Its connection is $A_3 = -2aI_3$, $A_2 = 3aI_2$, so both local curvature and all total-determinant periods cancel exactly.

The tensor-power overlap protocol gives block links with phases -2 and 3 times the line-link phase. For a loop of line comparison phase Φ ,

$$W_3 = e^{-2i\Phi} I_3, \quad W_2 = e^{3i\Phi} I_2.$$

The sign convention is that of endpoint-frame comparison, rather than forward state transport.

Proposition 19.2 (Central Wilson-defect ratio). *Define $D_m = m - \text{Re Tr } W_m$. Whenever $D_3 \neq 0$,*

$$\frac{D_2}{D_3} = \frac{2[1 - \cos(3\Phi)]}{3[1 - \cos(2\Phi)]} = \frac{3}{2} \left[1 - \frac{5}{12} \Phi^2 + O(\Phi^4) \right]. \quad (71)$$

Proof. The traces are $3 \cos(2\Phi)$ and $2 \cos(3\Phi)$. Expansion of each cosine gives the ratio and its first correction. $\square$

The limit $3/2$ follows from the rank-weight factors $18/12$, not from a particle charge measurement or a gauge-coupling ratio. The individual central module weights are not quark or lepton hypercharges. The ratio is undefined at denominator zeros, and finite loops need the exact formula rather than the small-loop limit.

For a latitude loop with $b = 1/4$ the continuum line phase is $\pi/2$. The exact trace deficits then give $D_3 = 6$, $D_2 = 2$, and $D_2/D_3 = 1/3$. The same family's infinitesimal-loop ratio is $3/2$. This contrast is a useful check that finite holonomy should not be mistaken for its curvature approximation.

19.3 Selector and phase-register timing laws

The carrier-selector semigroup has exact unwanted-selector decay $e^{-\gamma_m t}$, whereas separate volume coherences decay $e^{-18\gamma_\phi t}$ and the product coherence remains fixed. These rates belong to different channels and need not coincide. Measuring only one time scale cannot establish the other's normalization. Adding an independent total-phase noise channel would attenuate the product reference and invalidate the assumed determinant retention; its absence is therefore a testable implementation condition.

19.4 What would count as a physical prediction of SMM/DTT?

The present predictions concern specified finite models. A claim about nature requires an independently identified physical carrier, a map from laboratory or spacetime controls to $Z(x)$ or (ϑ, φ) , and a calibration of comparison spacing and response normalization. DTT must select these identifications rather than merely accommodate an observed phase after the fact. Distinguishing predictions from implementation checks is essential: the exact formulas make the model testable, but do not by themselves establish that nature implements it.

20 A finite joint record update: retained references and coherent return

We now give a completely specified, restricted microscopic update. Its purpose is to join completion and comparison in one experiment and to locate the reference resources that experiment consumes. It does not derive its graph, ranks, or hopping Hamiltonian from the source clock. The construction works for arbitrary positive carrier ranks; the earlier source/interface assumptions supply the particular ranks 3 and 2. The mathematical gain is a common singlet projector for the completion test and the binding penalty, an exact retained-algebra calculation, and a return statistic with a controlled remainder. The unavoidable preparation assumptions are stated separately.

20.1 Finite spaces, preparations, and the completion instrument

Take a supplied four-cycle with vertices $v = 0, 1, 2, 3$. Its record space is

$$\mathcal{D} = \bigotimes_v \mathcal{D}_v, \quad \mathcal{D}_v = \bigoplus_{a \in \{3,2\}} (L_{v,a} \otimes E_{v,a}), \quad \dim L_{v,a} = \dim E_{v,a} = a.$$

The left factors carry selectors. The right algebra is $\mathcal{B}_R = \bigotimes_v (M_3 \oplus M_2)$, represented with identity on every left factor. Thus $\dim_{\mathbb{C}} \mathcal{B}_R = 13^4$. Let $\mathcal{E}_{\text{rec}}$ first delete off-diagonal 3–2 blocks and then apply the channel of Definition 10.2 to each remaining left factor. It is the trace-preserving conditional expectation onto $\mathcal{B}_R$.

At each vertex attach a four-dimensional tensor-label space

$$\mathcal{R}_v = \mathbb{C}|00\rangle \oplus \det E_{v,3} \oplus \det E_{v,2} \oplus (\det E_{v,3} \otimes \det E_{v,2}). \quad (72)$$

Each summand is a line. Before any determinant reference is prepared, the allowed block-frame group is $U(3) \times U(2)$. Define a *relative volume-label noise* by the operator q_v with eigenvalues $(0, -1, 1, 0)$ on these four summands, and put

$$\mathcal{D}_\phi(X) = \frac{1}{3} \sum_{k=0}^2 Z^k X Z^{-k}, \quad Z = \exp(2\pi i q_v / 3). \quad (73)$$

This is exactly the charge-eigenspace pinching, not a finite-time approximation to a diffusion. Its fixed algebra is $M_2 \oplus \mathbb{C} \oplus \mathbb{C}$. The zero-charge block contains $|00\rangle$ and the product-volume line.

The charges in this definition are relative *label* charges; their equality and opposite signs are a stated noise assumption, not the physical carrier weights $(-2, 3)$. The same construction on ranks (r, s) preserves $\det E_r \otimes \det E_s$. It therefore does not select $3 + 2$ or their primitive central generator. Replacing q_v by $6q_v$ and the sampling angle $2\pi/3$ by $2\pi/18$ leaves Z unchanged and recovers the charge normalization of Section 10.2. The compensating change of angle is essential for this finite twirl.

For one block $m \in \{3, 2\}$ use the auxiliary space $\mathcal{K}_m$, root singlet s_m , and Hamiltonian $H_m(g)$ of (1)–(2). There are no hidden additional comparisons: the four unitary links $U_{j+1,j;m}$ are supplied relational data. The dual factor stays at the root and is counted as part of the apparatus. Put

$$P = |s_m\rangle\langle s_m|, \quad Q = I - P, \quad U_\tau = e^{-i\tau H_m(g)/\hbar}, \quad \mathcal{D}_P(X) = PXP + QXQ.$$

The full finite core algebra is $B(\mathcal{D} \otimes \bigotimes_v \mathcal{R}_v \otimes \mathcal{K}_m)$. The inner step is conjugation by U_τ on the probe and identity on the other factors. Completion is

$$\mathcal{E} = \mathcal{E}_{\text{rec}} \otimes \mathcal{D}_\phi^{\otimes 4} \otimes \mathcal{D}_P, \quad \Phi(X) = U_\tau^* \mathcal{E}(X) U_\tau \quad (74)$$

in the Heisenberg picture. This notation includes the identity factors in U_τ . All maps are completely positive and unital.

The associated probe instrument has Kraus operators $M_1 = PU_\tau$ and $M_0 = QU_\tau$. Its classical outcome records singlet return or failure. Discarding that outcome gives the probe part of (74); retaining its sequence gives, for an initial probe s_m ,

$$\Pr(r_1, \dots, r_n) = \|M_{r_n} \cdots M_{r_1} s_m\|^2. \quad (75)$$

Thus composition is explicit and time ordered. Fresh preparation of s_m after every observation is an alternative reset protocol, in which successive trials have independent conditional statistics for fixed links. That reset consumes a preparation resource and is not implicit in (75).

Every completion step has a finite unitary dilation. Left depolarization is obtained by swapping a selector with a fresh maximally mixed register; block dephasing and P/Q pinching use controlled phase gates and fresh two-state registers; (73) uses a fresh uniform three-state register controlling Z^k . The inner step and all these couplings are reversible on the enlarged system. Irreversibility enters when the registers are discarded and are excluded from future tests. No claim is made that the seven-cycle alone supplies these reservoirs.

20.2 The exact fixed algebra and the surviving shared reference

Theorem 20.1 (Fixed algebra of the finite joint cycle). *For every τ , the fixed algebra of (74) is*

$$\text{Fix } \Phi = \mathcal{B}_R \otimes (M_2 \oplus \mathbb{C} \oplus \mathbb{C})^{\otimes 4} \otimes \{P, U_\tau\}', \quad (76)$$

where the last commutant is in $B(\mathcal{K}_m)$. If $0 < \tau \text{ diam Spec } H_m < 2\pi\hbar$, the last factor equals $\{P, H_m\}'$. Every local and joint right-carrier observable is fixed pointwise. In particular every pre-existing right–right correlation, including a source-prepared cross-record pairing, survives.

Proof. $\mathcal{E}$ is a Hilbert–Schmidt orthogonal projection. If $X = U_\tau^* \mathcal{E}(X) U_\tau$, then $\|X\|_2 = \|\mathcal{E}(X)\|_2$, so $\mathcal{E}(X) = X$ and $[X, U_\tau] = 0$. The converse is immediate. The range of $\mathcal{E}$ is exactly the tensor product displayed, with the auxiliary factor $\{P\}' = \mathbb{C}P \oplus B(Q\mathcal{K}_m)$. Since U_τ acts only on that factor, intersection with its commutant gives (76). The no-alias condition makes $\lambda \mapsto e^{-i\tau\lambda/\hbar}$ injective on the finite spectrum, so H_m and U_τ have the same spectral projections. The entire right algebra lies in the resulting fixed algebra. $\square$

This gives an explicit version of Proposition 6.2, rather than a way around it. Erasing a shared pairing while fixing both full local right algebras is impossible. To remove that pairing one must change the admissible initial preparations, cease fixing some right observables, or retain the pairing as named relational information. The construction chooses the first option only when independent endpoint structural data are required: its reference-free preparation class has product maximally mixed right states, product block probabilities, and no source register entangled with distinct right carriers. Correlated right inputs are also legitimate states of the model; then their correlations remain part of its completed data. A common origin or common clock is not a proof of the reference-free preparation restriction.

The auxiliary factor can also be calculated, including the information it does *not* forget. Write $H_m = \Delta I - \Delta P - tA(W)$, where $A(W)$ is the unit-hopping ring adjacency after three links have been gauged to identity. For $t \neq 0$ define its cyclic subspace $\mathcal{C} = \text{span}\{A(W)^k s_m : k \geq 0\}$. For each distinct adjacency eigenvalue λ , let n_λ be its multiplicity on $\mathcal{K}_m$.

Proposition 20.2 (Residual auxiliary algebra). *The root singlet has nonzero projection onto every adjacency eigenspace, and*

$$\{P, H_m\}' \cong \mathbb{C}I_{\mathcal{C}} \oplus \bigoplus_{\lambda} M_{n_\lambda - 1}, \quad \dim_{\mathbb{C}}\{P, H_m\}' = 1 + \sum_{\lambda} (n_\lambda - 1)^2, \quad (77)$$

with zero-size summands omitted. Hence completion does not delete all auxiliary memory. For generic holonomy with $4m$ distinct adjacency eigenvalues, the dimension is $1 + 4m(m - 1)^2$. At $W = I$ it is $1 + 2(m^2 - 1)^2 + (2m^2 - 1)^2$.

Proof. For eigenphases θ_a of W , the adjacency eigenvalues are $2 \cos[(\theta_a + 2\pi k)/4]$, $k = 0, 1, 2, 3$, each with the spectator dual multiplicity m . In a simultaneous holonomy eigenbasis the singlet overlaps the matched carrier–dual mode with modulus $1/\sqrt{4m}$. Every distinct adjacency eigenspace is therefore met. Its one cyclic direction is its spectral projection of s_m ; the remaining $n_\lambda - 1$ directions form the reducing complement. Commutation with P forces $Xs_m = cs_m$. Commutation with A then gives $XA^k s_m = cA^k s_m$. Applying the same argument to X^* removes cross terms with $\mathcal{C}^\perp$. On that complement only commutation with A remains, giving the displayed matrix blocks. Finally $[X, P] = [X, H_m] = 0$ is equivalent to $[X, P] = [X, A] = 0$ when $t \neq 0$. At identity the adjacency multiplicities are $(m^2, 2m^2, m^2)$. $\square$

Equation (77) is the fixed algebra, not all nondecaying information. On $\mathcal{C}^\perp$ the channel is unitary conjugation: dark coherences between distinct adjacency eigenvalues persist with rotating phases. No convergence to a fixed-point projection is asserted for this joint cycle.

The probe and the protected right records are distinct factors. If the comparison Hamiltonian instead acts directly on the physical right carriers, their pointwise fixation must be recomputed; it does not follow from this theorem. Likewise the root dual is an explicit comparison resource throughout the inner walk. It is not an erased shared frame hidden in an unnamed environment.

20.3 Preparing a product volume consumes a reference

Pinching preserves a vacuum-to-product coherence but cannot create it from a diagonal register state. An explicit preparation makes that distinction measurable. At one vertex put $D = \det E_3 \otimes \det E_2$ and restrict the register to $\mathbb{C}|0\rangle \oplus D$. Supply a reservoir $S = \mathbb{C}|0_S\rangle \oplus D^*$. The canonical unit tensor in $D \otimes D^*$ defines a unitary V which swaps $|0, 0_S\rangle$ with that tensor and fixes the other two occupation sectors. It commutes with the full $U(3) \times U(2)$ action: both swapped vectors have trivial total character.

Proposition 20.3 (A covariant preparation and its reference cost). *For a chosen unit tensor $\tau \in D$, prepare $|+\tau^*\rangle_S = (|0_S\rangle + |\tau^*\rangle)/\sqrt{2}$. Then*

$$V(|0\rangle \otimes |+\tau^*\rangle_S) = \frac{|0\rangle + |\tau\rangle}{\sqrt{2}} \otimes |\tau^*\rangle_S. \quad (78)$$

The phase coherence has been transferred from the reservoir into the product-volume register. Conversely no covariant channel acting on an invariant initial register and invariant ancillary state can generate a nonzero vacuum-to-product coherence when D has its nontrivial determinant character.

Proof. The swap maps the first initial component to $|\tau, \tau^*\rangle$ and leaves $|0, \tau^*\rangle$ fixed, which proves (78). This definition is independent of the phase used to write the canonical tensor $\tau \otimes \tau^*$. A covariant channel maps invariant inputs to invariant outputs. The off-diagonal operator $|\tau\rangle\langle 0|$ transforms by $\det g_3 \det g_2$, a nontrivial character, so its expectation in an invariant output is zero. Tracing out a covariantly coupled invariant reservoir does not change that conclusion. $\square$

Equivalently the swap is a calibrated pulse generated on the neutral two-dimensional sector by a Hamiltonian coupling vacuum to the canonical $D \otimes D^*$ tensor. Its pulse area and the coherent reservoir preparation are resources. Starting instead from both vacua, a partial pulse can make a globally invariant superposition of $|0, 0_S\rangle$ and $|\tau, \tau^*\rangle$, but its reduced register has zero vacuum-to-product coherence. Entanglement with a retained dual reference is not a standalone prepared determinant reference.

After the actual preparation (78), (73) preserves the product coherence. The stabilizer of the prepared splitting, metrics, and tensor is $S(U(3) \times U(2))$, as proved earlier. The channel has not generated this tensor from symmetric data, and no target group was required to define the enlarged covariant swap. Nevertheless choosing precisely this tensor resource and correlated relative noise is an additional physical choice. Global patching of tensors at different vertices is also an input; finite local preparation does not prove bundle descent.

20.4 One coherent return gives a calculated completion statistic

The same projector P now has two declared roles: return is the recorded completion outcome and ΔQ is its energetic failure penalty. The equality of this test and this penalty is the microscopic identification being proposed. The positive calibration Δ and the hopping terms remain assumptions. Under uninterrupted coherent evolution they give the static energy results of Section 2; a single subsequent completion measurement gives the following independent consequence.

Theorem 20.4 (Holonomy resolved by a coherent completion event). *Let $p_W(\tau) = |\langle s_m, e^{-i\tau H_m(W)/\hbar} s_m \rangle|^2$ and put $u = \tau/\hbar$, $L = \Delta + 4|t|$. For arbitrary unitary W ,*

$$p_I(\tau) - p_W(\tau) = \frac{t^4 u^4}{6} \left[1 - \frac{1}{m} \operatorname{Re} \operatorname{Tr} W \right] + R_W(u), \quad |R_W(u)| \leq \frac{(|u|L)^6 e^{|u|L}}{360}. \quad (79)$$

The estimate is uniform in W and includes central holonomies. For independent experiments on both blocks, with the same time τ , the leading small-curvature coefficients obey the two-block relation with $b_m = t_m^4 u^4 / (6m)$ and $b_1 = 12b_3 + 18b_2$. These dimensionless coefficients describe a completion probability, not a gauge coupling or a Hamiltonian stiffness.

Proof. Write $\mu_k = \langle s_m, H_m^k s_m \rangle$. Position parity and $P s_m = s_m$ give

$$\mu_1 = 0, \quad \mu_2 = 2t^2, \quad \mu_4 = 2\Delta^2 t^2 + t^4 \left[6 + \frac{2}{m} \operatorname{Re} \operatorname{Tr} W \right].$$

For μ_4 , the term with two transfers and two penalties is $2\Delta^2 t^2$; odd transfer paths cannot close on the bipartite cycle. Four transfers have six backtracking paths and two oriented cycles, whose singlet contractions are $\operatorname{Tr} W/m$ and its conjugate. Expanding the survival probability gives

$$p_W(u) = 1 - \mu_2 u^2 + (\mu_4/12 + \mu_2^2/4) u^4 + O(u^6).$$

The distribution of spectral energy differences in a survival probability is symmetric, so its odd powers vanish. Every energy difference has modulus at most $\text{diam Spec } H_m \leq \Delta + 4|t| = L$. Taylor's estimate for each spectral exponential bounds its remainder after degree five by $(|u|L)^6 e^{|u|L}/6!$; the spectral weights sum to one. Subtracting the two probabilities doubles that bound. The Δ -dependent and common backtracking contributions cancel, giving (79). Finally $1 - \text{Re Tr } e^{iX}/m = \text{Tr } X^2/(2m) + O(\|X\|^4)$, and the central weights $(-2, 3)$ give twelve and eighteen. $\square$

The phase sensitivity depends on coherent alternatives *before* completion. A different, forced-circuit protocol first transports the carrier once around the loop and then tests only the root singlet. Its success probability is

$$p_{\text{circuit}}(W) = |\text{Tr } W/m|^2. \quad (80)$$

For $W = e^{i\theta} I_m$ this is one for every θ , although (79) resolves $1 - \cos \theta$ at fourth order. Thus a single prescribed circuit followed by singlet completion cannot replace the coherent waiting, backtracking, and circulation alternatives in the given Hamiltonian. Frequent intermediate P/Q measurements also change its dynamics and may cause a Zeno limit. The static or effective Hamiltonian theorems apply to the specified coherent evolution, not automatically to that repeatedly measured limit.

20.5 Exactly what this construction improves

Before relational links are supplied, the reference-free endpoint preparation, intrinsic root singlet, and product-volume tensors select no cross-record defining-carrier map: they admit independent adapted endpoint presentations. The obstruction of Theorem 6.5 therefore applies to those endpoint data. The links supplied to this experiment are the missing relational variables. A fixed collection of links or a prepared right pairing need not be invariant under independent *active* frame rotations; independent passive presentation changes still transform all of these data covariantly. The distinction is necessary to avoid describing a retained common reference as erased.

Ingredient	Status in the joint update
Inner evolution and completed output	SMM/DTT motivation; represented here by a unitary step followed by an explicit instrument. Neither ontology selects the listed finite operators.
Ranks, graph, and allowed links	Supplied microscopic assumptions, with ranks obtained only under the earlier source/interface axioms. Adjacency and unitary comparison are not generated by this update.
Selector reservoirs and phase pinching	Independently specified microscopic noise and environment-reset assumptions; their fixed algebras are derived.
No shared right reference initially	Preparation restriction, compatible with a common source but not entailed by it. Removing the restriction leaves any prepared joint reference observable.
Product-volume reference	Prepared from an explicit coherent dual-line reservoir and subsequently preserved; not created by symmetric noise.
Singlet, hopping, and ΔQ	Local probe preparation and microscopic Hamiltonian assumptions. The same P defines binding and completion; its return law and static loop response are calculated consequences.

Retained algebra and comparisons	Exact fixed-algebra theorem, survival of all right correlations, residual auxiliary algebra, and coherent holonomy-dependent outcome probabilities.
Gauss sector, dimension, and matter	Not selected. A gauge-covariant instrument is not amplitude sewing; no adjacency growth, diffusion, pointlike magnetic source, CAR, even-sector choice, or physical Weyl chirality is derived. M1–M4 and the dimensional hypotheses remain additional assumptions.

This is a partial microscopic unification of record completion and singlet comparison, together with a preparation obstruction. It does not turn the independent tensor resource, electric Laplacian, physical chirality, or spatial continuum into consequences of the temporal ontology. Its exact fixed algebra specifies the missing estimates and selection problems more narrowly than an unspecified joint-update proposal.

21 From gauge geometry to Yang–Mills dynamics

Gauge covariance does not make a background Berry connection a propagating field. To obtain interactions, the retained sectors must have physical fluctuations whose connection variations survive the low-energy limit. A microscopic derivation must establish their kinetic term, constraints, and coupling to physical matter.

Proposition 21.1 (Conditional leading curvature action). *Assume an emergent four-dimensional Lorentzian metric, a physically variable connection, a local analytic derivative expansion, gauge and Lorentz covariance, no preferred background tensors, and a nondegenerate positive parity-even quadratic kinetic term. At quadratic order in curvature, the leading pure-gauge action is a sum of Yang–Mills terms, one for each simple ideal and the Abelian factor, up to independent coefficients. Parity-odd topological terms may also occur.*

Proof. F transforms by conjugation. A quadratic local scalar contracts spacetime indices with the metric and Lie-algebra indices with an invariant bilinear form. On each compact simple ideal that form is unique up to scale; invariant cross terms between inequivalent simple ideals and the center vanish. The one-dimensional center admits its own coefficient. Contraction with the oriented volume tensor gives the parity-odd terms. Terms with additional derivatives or curvature powers occur at higher order in the assumed expansion. $\square$

For anti-Hermitian curvatures use $\langle X, Y \rangle = -\text{Tr}(XY)$ on the simple factors and a fixed positive normalization on the center. In signature $(-+++)$ the parity-even action is

$$S_{\text{kin}} = -\frac{1}{4} \int_X \sqrt{-g} d^4x \left[\frac{\langle F_3, F_3 \rangle}{g_3^2} + \frac{\langle F_2, F_2 \rangle}{g_2^2} + \frac{f_{\mu\nu} f^{\mu\nu}}{g_1^2} \right],$$

where the first two contractions include spacetime indices. The three positive coefficients are independent; no coupling unification follows. The central generator $i \text{diag}(-2I_3, 3I_2)$ fixes the integral normalization inherited by the matter module in Section 24; its physical identification still requires M1–M4 and a coupling normalization.

For a matter representation ρ , covariance permits $D_\mu = \partial_\mu + \rho_*(A_\mu)$; it does not select ρ . Gauss constraints require a stated physical state-space or action principle, such as the coherent sewing rule of Section 12; they do not follow from passive frame freedom alone. Anomaly cancellation is an independent requirement for any proposed chiral matter content.

21.1 Retained carriers and removable completion sectors

A physical retained carrier can support observable transport. A removable auxiliary completion sector must instead leave no unintended finite-energy gauge field. Rank one alone does not ensure this: a line bundle can have nonzero Abelian curvature or flat holonomy. Any residual band connection must be proved unobservable, identified with already declared physical data, or decoupled with its responses controlled. A finite inner-cycle gap is not a four-dimensional Yang–Mills mass gap.

21.2 What the microscopic models settle and leave open

The models in Sections 2 and 3 settle an existence question: specified incomplete-transfer dynamics can generate loop energy, and the non-Abelian construction is compatible with the retained $3 + 2$ group. Quantization on a compact link group with a positive electric Laplacian supplies an autonomous finite-volume system. This is a stronger result than curvature accessibility under externally chosen controls. It is not yet a derivation of those quantum links or their Hamiltonian from the original finite source.

The joint update in Section 20 now gives an explicit partial implementation of generation, completion, and comparison tests. The remaining benchmark is to select that update and its reference resources independently, generate adjacency, and establish a stable spatial and temporal scaling regime. The bounded-electric theorem gives controlled finite-volume dynamics, while the uncapped comparison still requires the weighted electric-tail estimate (22). Thermodynamic and continuum limits, physical auxiliary decoupling, and a chiral matter coupling remain separate tasks. A flat vacuum is compatible with gauge dynamics; the excluded alternative is a model permitting only flat transport and no physical gauge fluctuations.

Equation (14) supplies the explicit finite-model realization of the earlier schematic target

$$H_{\text{eff}} = \sum_{e,a} \alpha_a \|\mathcal{E}_e^{(a)}\|^2 + \sum_p V(W_{p,3}, W_{p,2}) + H_{\text{matter}} + \cdots, \quad \mathcal{G}_v|\text{phys}\rangle = 0. \quad (81)$$

Its genuine chiral matter term is still to be constructed. Neither the primitive static relation $\kappa_1 = 12\kappa_3 + 18\kappa_2$ nor a fixed carrier gap establishes equality of temporal kinetic coefficients, a universal propagation speed, confinement, or a four-dimensional continuum Yang–Mills mass gap.

22 Seven-cycle response and perturbation budget

Definition 22.1 (Common physical normalization). *Let one completed source step have duration τ . Define the two response operators with the same coefficient $\hbar/\tau$:*

$$G_3 = \frac{\hbar}{\tau}|I - K_3|, \quad K_3 = K|_{E_3}, \quad G_2 = \frac{\hbar}{\tau}|D_J|.$$

The common factor cancels in every dimensionless gap ratio. Here the defect response is a protecting excitation gap outside its kernel: $G_2|_{E_2} = 0$. It is not an intrinsic doublet energy or a particle mass.

Theorem 22.2 (Exact source response). *The lowest nonzero triplet response and defect response satisfy*

$$\frac{\text{gap}(G_2)}{\text{gap}(G_3)} = \mathfrak{r}_7 = \frac{1}{2 \sin(\pi/7)} = 1.152382435481243 \dots$$

With the original adjacent-frequency labeling, the recurrence also gives the internal chord ratio

$$\mathfrak{p}_7 = \frac{\sin(2\pi/7)}{\sin(\pi/7)} = 2 \cos(\pi/7) = 1.801937735804838 \dots$$

Both are independent of a Standard-Model finite geometry.

Theorem 22.3 (Rigidity within the stipulated response class). *Any correction restricted to the exact response definitions in Definition 22.1, the regular-cycle relations, the marked cut, and a single common clock coefficient changes (K, Ω) only by unitary equivalence, a discrete automorphism of $\mathbb{Z}_7$, and common multiplicative normalization. Such corrections leave $\mathbf{r}_7$ and, when adjacent-frequency labeling is preserved, $\mathbf{p}_7$ exactly invariant.*

Proof. Every real regular representation of $\mathbb{Z}_7$ with a marked orthonormal orbit is unitarily equivalent to the shift representation. Automorphisms permute the nonzero characters. A common coefficient cancels in ratios. General symmetry-preserving effective corrections, such as spectral functions of K or independent block terms, are outside this restricted response class. No radiative nonrenormalization theorem is asserted. $\square$

No-go theorem 22.4 (Generic infrared mass-ratio claim). *The group $S(U(3) \times U(2))$ alone permits independent gauge-invariant renormalizations of the triplet and doublet blocks. Therefore $\mathbf{r}_7$ cannot be advertised as an infrared particle-mass ratio unless an additional Ward identity proves equality of the two block renormalizations. The theorem predicts a source-transfer response, not a Standard-Model mass.*

22.1 Uncertainty budget

Put $s = 2 \sin(\pi/7)$. If $|\hat{\Delta}_2 - 1| \leq \varepsilon_2$ and $|\hat{\Delta}_3 - s| \leq \varepsilon_3 < s$, then

$$|\hat{r} - r_7| \leq \frac{s\varepsilon_2 + \varepsilon_3}{s(s - \varepsilon_3)}, \quad (82)$$

$$\left| \frac{\hat{r}}{r_7} - 1 \right| \leq \frac{s\varepsilon_2 + \varepsilon_3}{s - \varepsilon_3}. \quad (83)$$

For equal errors $\varepsilon_2 = \varepsilon_3 = 10^{-3}$, the relative bound is 2.155×10^{-3} . Common timing and energy calibration cancel exactly. A measured block-dependent normalization mismatch $\eta = Z_2/Z_3 - 1$ contributes an additional relative systematic $|\eta|$.

Falsifier 22.5 (Direct finite-source test). Implement the seven-state shift and its marked cut in a photonic walk, trapped-ion qudit, superconducting qudit, or exact numerical transfer matrix. Use one calibrated step time for both blocks. Process tomography must show:

- (a) first return at seven and no hidden orbit state;
- (b) a three-line positive-frequency sector;
- (c) a two-dimensional doubled cut-defect kernel;
- (d) fixed-algebra dimension $9 + 4 = 13$ after record mixing;
- (e) no designated retained right in/out polarization and no extra fixed selector algebra;
- (f) $\hat{r}$ within (82).

Failure rejects this finite implementation under the stated calibration and record assumptions. Implementing a designed seven-state clock is a check of the model, not evidence that nature realizes SMM/DTT.

23 Robustness and no-go summary

Statement	Consequence
Frobenius rigidity	The source class has only the quaternionic noncommutative branch.
Minimal quadrature	Orders below seven are incomplete; orders above seven contain spectators.
Octonionic alternative	If one selects the octonionic source and retains the same quadrature rule, the order is fifteen; nonassociativity alone does not force that branch.
K -alone marker obstruction	A source/reset interface is logically necessary; a closed operator cannot generate a noninjective record expectation.
Projector-erasure no-go	Selector projectors must act in a complementary algebra; standard form provides the canonical complement.
Polarization obstruction	A designated retained right polarization reduces the doublet stabilizer to $U(1)^2$.
Curvature-only no-go	Zero determinant curvature does not control flat periods; the global wedge tensor does.
No-extra-factor theorem	For the stipulated record model, selector mixing and block dephasing fix the declared carrier algebra; the retained total wedge leaves one relative $U(1)$.
Frame/holonomy distinction	The conditional frame stabilizer is fixed; actual source holonomy requires a curvature calculation.
Mass-ratio no-go	A source spectral ratio is exact, but an infrared mass interpretation needs a separate nonrenormalization theorem.

24 Focused extension: one-generation matter reconstruction

The preceding construction provides an internal $3 + 2$ carrier and transport, not fermions. We now ask whether a single, explicitly specified matter functor supplies a complete generation without separately assigning its six multiplet charges. The answer is affirmative under a new exterior-completion principle, together with an independent spacetime Weyl-field assumption. This is a conditional representation reconstruction, not a proof that these assumptions are realized in nature.

24.1 Additional matter principles and their temporal motivation

Set $E = E_3 \oplus E_2$, with $\text{rank}_{\mathbb{C}_J} E = 5$ and retained unit tensor $\tau \in \det E$. The use of $\mathbb{C}_J$ continues to mean the selected ordinary complex phase algebra, not a split-complex temporal unit. Add the following principles:

- (M1) **Exterior composition.** Internal matter labels form the cyclic vacuum module of five canonical anticommuting creation operations associated with E . No independent flavor multiplicity space or additional elementary internal label is introduced.
- (M2) **Vacuum-connected pair completion.** The retained internal matter module is the full sector reachable from the distinguished degree-zero vector by arbitrary pair creation; internal odd degree is excluded from the retained module. Closure includes every even-degree monomial, not just a chosen list of representations.

- (M3) **Spacetime field interpretation.** Each retained internal component labels an independent left-handed Weyl field on an assumed four-dimensional oriented spin spacetime. A conjugate internal module is not added as another set of left-handed fields. Antiparticles are those of these quantized fields, rather than an extra mirror generation.
- (M4) **Inherited gauge action.** Matter transforms only by the exterior lift of the source-compatible G_{src} action. There is no independent central character twist, degree rephasing gauge field, or extra gauge factor.

SMM's common source motivates reuse of E rather than independent charge labels. Pair completion is a proposed interpretation of a conjugate-completion interface. Neither motivation proves CAR, selects even internal degree, or establishes Lorentzian chirality. M1–M4 are new physical restrictions. Their advantage is that, once stated, the multiplet list and relative hypercharges are calculated together. Their selection is the principal unresolved physical problem of this extension.

Proposition 24.1 (Cyclic exterior module and retained rank). *The finite cyclic CAR vacuum module is ΛE of dimension 32. Under M2 its retained module is*

$$\mathcal{M}_+ = \Lambda^{\text{even}} E = \Lambda^0 E \oplus \Lambda^2 E \oplus \Lambda^4 E, \quad \dim_{\mathbb{C}_J} \mathcal{M}_+ = 1 + 10 + 5 = 16.$$

Proof. For an orthonormal basis e_i , the creation operators obey $a_i^\dagger a_j^\dagger = -a_j^\dagger a_i^\dagger$ and the annihilation operators obey $\{a_i, a_j^\dagger\} = \delta_{ij}$. Applied to a normalized vacuum annihilated by all a_i , the ordered monomials for all subsets of $\{1, \dots, 5\}$ form an orthonormal basis. They identify the module with ΛE . Every even monomial is a product of pair creators, and every product of pair creators has even degree; dimensions are the binomial coefficients shown. $\square$

Pair reachability here is an algebraic closure statement, not time evolution under a charge-conserving Hamiltonian. Individual pair creators can carry gauge charge; realizing such transitions physically requires specified fields or an environment exchanging that charge. This finite internal CAR construction does not derive the spin–statistics theorem. Its degree-zero vector is an internal label, not the physical no-particle vacuum of a spacetime quantum field theory. M3 assigns a Weyl field even to that component. Internal exterior degree and physical Fermi parity are different notions.

24.2 Primitive central generator and quotient compatibility

Proposition 24.2 (Inherited central charges). *The primitive integral central circle of the source-compatible action on E is, up to orientation,*

$$g(\theta) = \text{diag}(e^{-2i\theta} I_3, e^{3i\theta} I_2), \quad \theta \sim \theta + 2\pi.$$

The charge operator on an exterior monomial containing p triplet factors and r doublet factors is

$$q(p, r) = -2p + 3r.$$

In the conventional hypercharge normalization $Y = q/6$, all relative hypercharges of $\mathcal{M}_+$ are fixed.

Proof. A scalar central action has integer weights (a, b) on the two blocks. Retention of τ requires $3a + 2b = 0$. Coprime nonzero solutions are $(a, b) = (-2, 3)$ and its sign reversal. Exterior multiplication adds weights, giving the formula. The choice of orientation identifies the branch called particles. Division by six is a convention fixed below by the neutral component of a weak Higgs doublet; it does not determine a gauge coupling. $\square$

A determinant twist is trivial on G_{src} . Other character twists of the matter module are not excluded by the group alone; M4 excludes them by requiring the unmodified exterior action.

Write the covering map as

$$(A, B, z) \mapsto \text{diag}(z^{-2}A, z^3B).$$

Its kernel is generated by $(e^{2\pi i/3}I_3, -I_2, e^{i\pi/3})$. A representation with color triality t_3 , weak center parity t_2 , and integer central charge q descends to the quotient precisely when

$$q + 2t_3 + 3t_2 = 0 \pmod{6}. \quad (84)$$

Every exterior sector satisfies this condition: $t_3 = p \pmod{3}$, $t_2 = r \pmod{2}$, and $q = -2p + 3r$. Thus quotient descent is inherited without extra charge assignments. Conversely, the congruence alone allows infinitely many charges and does not determine the Standard Model spectrum.

24.3 One generation as an explicit decomposition

Theorem 24.3 (Conditional one-generation reconstruction). *Under the earlier 3+2 source/record construction and M1–M4, the retained internal module decomposes as*

$$\begin{aligned} \mathcal{M}_+ \cong & (\mathbf{3}, \mathbf{2})_{1/6} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-2/3} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{1/3} \\ & \oplus (\mathbf{1}, \mathbf{2})_{-1/2} \oplus (\mathbf{1}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{1})_0. \end{aligned} \quad (85)$$

With M3 these are one Standard Model generation in all-left-handed notation, together with a sterile conjugate-neutrino field. Each multiplet has multiplicity one. No separate hypercharge parameter is assigned to any multiplet.

Proof. Use $\Lambda^k(E_3 \oplus E_2) = \bigoplus_{p+r=k} \Lambda^p E_3 \otimes \Lambda^r E_2$. In degree two, $(p, r) = (2, 0), (1, 1), (0, 2)$ give respectively the color antitriplet, quark doublet, and charged singlet. In degree four, $(2, 2), (3, 1)$ give the remaining antitriplet and lepton doublet. Degree zero gives a singlet. Their weights follow from Proposition 24.2. The $SU(3)$ identification $\Lambda^2 \mathbf{3} \cong \bar{\mathbf{3}}$ and the $SU(2)$ identification $\Lambda^2 \mathbf{2} \cong \mathbf{1}$ yield the stated decomposition. $\square$

Field	Internal sector	$SU(3) \times SU(2)$	Y	q	Dimension
Q	$E_3 \otimes E_2$	$(\mathbf{3}, \mathbf{2})$	$1/6$	1	6
u^c	$\Lambda^2 E_3$	$(\bar{\mathbf{3}}, \mathbf{1})$	$-2/3$	-4	3
d^c	$\Lambda^2 E_3 \otimes \Lambda^2 E_2$	$(\bar{\mathbf{3}}, \mathbf{1})$	$1/3$	2	3
L	$\Lambda^3 E_3 \otimes E_2$	$(\mathbf{1}, \mathbf{2})$	$-1/2$	-3	2
e^c	$\Lambda^2 E_2$	$(\mathbf{1}, \mathbf{1})$	1	6	1
ν^c	$\Lambda^0 E$	$(\mathbf{1}, \mathbf{1})$	0	0	1

Table 9: Calculated representation content under M1–M4 for $E = \mathbb{C}^5$ with defining integral charges $(-2, -2, -2, 3, 3)$. The even exterior sectors have dimensions $1 + 10 + 5 = 16$. Superscript c denotes a left-handed charge-conjugate field, not an additional right-handed Weyl field. These are calculated model outputs, not observed particle data.

The determinant tensor gives $\Lambda^4 E \cong E^*$, so the same result is $\mathbf{1} \oplus \Lambda^2 \mathbf{5} \oplus \bar{\mathbf{5}}$ under the algebraic $SU(5)$ embedding. This is the familiar $\mathbf{1} \oplus \mathbf{10} \oplus \bar{\mathbf{5}}$ packaging and the even half-spin construction used in Spin(10) representation theory [1]. We do not claim a new exterior-algebra origin of the generation. The proposed extension connects that established matter functor to the previously constructed temporal carriers, with explicit hypothesis and verification boundaries. Neither off-block $SU(5)$ gauge bosons nor Spin(10) gauge dynamics are introduced. The retained splitting keeps the gauged group G_{src} .

24.4 Chirality, mirrors, and a finite completion selector

Proposition 24.4 (Odd parity is the conjugate internal module). *The retained tensor and metric identify $\Lambda^{\text{odd}} E$ with the complex-conjugate representation of $\mathcal{M}_+$. Keeping both modules as independent left-handed Weyl fields gives vectorlike mirror pairs, whereas M3 keeps only $\mathcal{M}_+$ as independent left-handed fields.*

Proof. The perfect wedge pairing $\Lambda^k E \otimes \Lambda^{5-k} E \rightarrow \det E$ and τ identify complementary degrees as dual representations. Since five is odd, they exchange parity. The Hermitian metric identifies the dual with the conjugate representation. A left-handed field in a representation and an independent left-handed field in its conjugate can admit an invariant Dirac mass pairing. $\square$

An internal parity selector can be displayed without assigning individual multiplet energies. Let N be exterior degree, $\Pi = (-1)^N$, and

$$H_{\text{pair}} = \Delta \frac{I - \Pi}{2}, \quad \Delta > 0.$$

It has a rank-sixteen zero-energy even sector and a rank-sixteen odd sector at Δ , and commutes with the lifted gauge action. Pair creation and annihilation preserve this splitting and connect the vacuum to all even sectors. This is a finite internal Hamiltonian implementing M2. It supplies neither a Lorentzian Weyl operator nor a regulator for a chiral gauge theory. In particular a gauge-invariant internal penalty is not, by itself, a local Lorentz-invariant mass term that removes a mirror generation. A physical mirror-decoupling mechanism and spacetime chirality remain independent problems. The proposal deliberately avoids identifying exterior parity with left/right spacetime chirality.

24.5 Anomaly checks and their scope

Theorem 24.5 (Perturbative anomaly cancellation and the usual weak global test). *The left-handed module in Theorem 24.3 cancels the $SU(3)^3$, $SU(3)^2 U(1)$, $SU(2)^2 U(1)$, $U(1)^3$, and mixed gravitational- $U(1)$ anomalies. It also has an even number of fundamental weak doublets and passes the usual $SU(2)$ Witten test on spin backgrounds.*

Proof. Normalize the color fundamental cubic coefficient to one and ignore common nonzero factors in mixed indices. Using integral charges, the anomaly coefficients are

$$\mathcal{A}_{333} = 2 - 1 - 1 = 0, \tag{86}$$

$$\mathcal{A}_{33q} = 2(1) + (-4) + 2 = 0, \tag{87}$$

$$\mathcal{A}_{22q} = 3(1) + (-3) = 0, \tag{88}$$

$$\mathcal{A}_{q \text{ grav}} = 6(1) + 3(-4) + 3(2) + 2(-3) + 6 + 0 = 0, \tag{89}$$

$$\mathcal{A}_{qqq} = 6(1)^3 + 3(-4)^3 + 3(2)^3 + 2(-3)^3 + 6^3 + 0 = 0. \tag{90}$$

The weak doublet count is $3 + 1 = 4$. The weak cubic perturbative anomaly vanishes because its representations are pseudoreal; mixed terms with a single simple-group generator vanish by tracelessness. Replacing q by $Y = q/6$ rescales zero coefficients to zero. $\square$

The familiar weak global anomaly is discussed in [3]. These checks establish the displayed anomaly tests for the proposed module. They are not a complete analysis of every quotient-group bundle on arbitrary non-spin backgrounds, a construction of the quantum chiral measure, or an existence proof for the interacting continuum theory. No anomaly computation derives M1–M3.

24.6 Why anomalies alone do not select these charges

A useful adversarial test is to forget the inherited exterior action. Keep the same six non-Abelian representations, with arbitrary Abelian charges, and allow Dirac Yukawa interactions with one doublet of charge h . Writing $x = q_Q$ and $\ell = q_L$, invariance requires

$$q_u = -x - h, \quad q_d = -x + h, \quad q_e = -\ell + h, \quad q_\nu = -\ell - h.$$

The weak mixed anomaly imposes $\ell = -3x$. The color mixed and gravitational anomalies then vanish identically, and the cubic anomaly is

$$\mathcal{A}_{qqq} = -6h^2(3x + \ell) = 0.$$

Thus this enlarged ansatz retains two charge parameters (x, h) rather than a unique hypercharge assignment. Choosing $(x, h) = (1, 3)$ recovers the inherited integral charges. This ambiguity with a neutral-fermion extension is established in the charge-quantization literature [2].

For the present spectrum the corresponding $B - L$ charges are

$$(1/3, -1/3, -1/3, -1, 1, 1)$$

for $(Q, u^c, d^c, L, e^c, \nu^c)$. They and Y span the continuous charge solutions above; the mixed cubic coefficients $\text{Tr } Y^2(B - L)$ and $\text{Tr } Y(B - L)^2$ both vanish. Arbitrary real combinations need not exponentiate to a compact group on the declared charge lattice, but the local anomaly equations alone do not fix that lattice or the inherited action.

The exterior principle removes this freedom by fixing the action on all degrees simultaneously and making $\Lambda^0 E$ neutral. In the ansatz above, $q_\nu = 0$ gives $h = 3x$, leaving only overall normalization; the primitive central circle fixes that normalization in integral units. Alternatively a nonzero Majorana term for ν^c requires $2q_\nu = 0$ for a continuous $U(1)$ and has the same local effect. The latter is an additional interaction assumption, not an explanation for why the exterior functor is selected. Anomaly cancellation validates the reconstructed representation; it does not independently reconstruct it.

24.7 Minimal Higgs interface and permitted interactions

To discuss electric charge and masses, add a scalar section $H \in E_2$, without adding a color-triplet scalar. This is an extra scalar-sector assumption. Its integral charge is 3, hence $Y_H = 1/2$. If a nonzero vacuum expectation value lies in the $T_3 = -1/2$ component, the generator leaving it invariant is

$$Q_{\text{em}} = T_3 + \frac{q}{6} = T_3 + Y.$$

It gives $Q = (2/3, -1/3)$, $L = (0, -1)$, and singlet charges $u^c = -2/3$, $d^c = 1/3$, $e^c = 1$, $\nu^c = 0$. This reproduces charge assignments conditional on the Higgs interface and breaking direction. No Higgs potential, nonzero vacuum, or mass scale is derived.

Proposition 24.6 (Gauge-compatible Yukawa interfaces). *For this scalar interface the invariant trilinear structures include*

$$Qu^c H, \quad Qd^c H^*, \quad Le^c H^*, \quad L\nu^c H.$$

A singlet Majorana bilinear $\nu^c \nu^c$ is also allowed. Their coefficients are independent parameters under G_{src} .

Proof. The respective integral charge sums are $1 - 4 + 3$, $1 + 2 - 3$, $-3 + 6 - 3$, and $-3 + 0 + 3$, all zero. Color indices contract a triplet with an antitriplet; weak indices use the invariant alternating tensor or the dual pairing. Their intrinsic origin can be written using τ : the up

interface is $\Lambda^2 E \otimes \Lambda^2 E \otimes E \rightarrow \det E$ by wedge multiplication. The down and charged-lepton interfaces use $\alpha \wedge \iota_{H^*} \beta$ for $\alpha \in \Lambda^2 E$, $\beta \in \Lambda^4 E$. The neutrino interface wedges $\Lambda^4 E$ with $H \in E$ and multiplies by the degree-zero singlet. Restricting H to E_2 gives the listed channels. Lorentz spinor contractions use M3. $\square$

Corollary 24.7 (Unbroken-phase mass obstruction). *For a single retained generation, the only gauge-invariant Weyl mass bilinear without extra scalar fields is the sterile $\nu^c \nu^c$ term. None of the five charged multiplets has a conjugate partner in $\mathcal{M}_+$.*

Proof. A mass bilinear must have total central charge zero. Among the integral charges 1, -4, 2, -3, 6, 0, the only zero sum of two entries is $0 + 0$. The sole zero-charge component is the gauge singlet, so it admits the symmetric internal bilinear needed for a single two-component Majorana field. This is an unbroken-phase selection rule, not a calculation of its coefficient. $\square$

A common wedge notation does not force equal Yukawa coefficients: the retained 3+2 splitting allows invariant sector projections and independent coefficients. No $SU(5)$ mass relation, proton-decay rate, or coupling unification follows. Gauge symmetry also permits higher-dimensional baryon/lepton-number violating interactions, such as $QQQL$ and $u^c u^c d^c e^c$; their integral charge sums vanish. Their absence would need an extra physical restriction. The construction is not a phenomenological model until interaction strengths and scales are determined.

24.8 Quaternionic spin lift as a geometric interface

The following is a geometric interface, not a derivation of spin-1/2, CAR, or a physical Weyl field. The quaternionic source offers this interface once an operational spatial coframe has been constructed. Unit quaternions act on $W = \text{Im } \mathbb{H}$ by $w \mapsto qwq^{-1}$. This map from $SU(2)$ onto $SO(3)$ has kernel $\{1, -1\}$: a quaternion commuting with all imaginary quaternions is real, and a real unit is ± 1 . For a unit imaginary axis u ,

$$q(\theta) = \cos(\theta/2) + u \sin(\theta/2)$$

implements a spatial rotation by θ . It ends at -1 after 2π and returns to $+1$ after 4π . A nontrivial loop in the oriented spatial frame space therefore has the familiar sign-changing spin lift. This explains what a temporal orientation cycle would have to realize to support a spinorial response.

It does not show that every source cycle is that nontrivial loop, that the physical carrier transforms in the spin-1/2 representation, or that its excitations obey CAR. The spatial spin lift acts on the spacetime spinor factor. The weak $SU(2)$ acts on the independently retained defect factor. Their equal abstract group type does not identify their physical actions. M3 still requires a four-dimensional spin structure and a Weyl-field interpretation; internal exterior parity is not spacetime chirality.

24.9 A precise test for any proposed three-family extension

A natural complex surface associated with a three-dimensional complex space V is its ray space $\mathbb{CP}^2 = \mathbb{P}(V)$. If one *chooses* the positive line bundle $\mathcal{O}(1)$, its global holomorphic sections are the linear homogeneous polynomials on V , so

$$H^0(\mathbb{P}(V), \mathcal{O}(1)) \cong V^*, \quad \dim = 3. \quad (91)$$

For comparison, the canonical spin- c Dolbeault index is

$$\text{ind } \bar{\partial}_{\mathcal{O}(1)} = \int_{\mathbb{CP}^2} e^h \text{Td}(T\mathbb{CP}^2) = \int (1 + h + h^2/2)(1 + 3h/2 + h^2) = 3,$$

where $h = c_1(\mathcal{O}(1))$ and $\int h^2 = 1$. The Euler sequence gives $c_1(T) = 3h$, $c_2(T) = 3h^2$, hence the displayed Todd class. The index formula is standard [34]; the direct section count already establishes the three holomorphic modes for this specified bundle. A bare index controls a difference of chiral kernel dimensions, not the exact kernel under arbitrary perturbations. Metric, operator, bundle, and stability assumptions are necessary when interpreting it as a protected low-energy spectrum.

Proposition 24.8 (The color carrier cannot be counted again as three neutral families). *If V in (91) is the same gauge triplet E_3 used in the reconstructed generation, its section space carries the dual color action, not a trivial flavor action. Tensoring that space with $\mathcal{M}_+$ does not give three gauge-identical generations.*

Proof. For a linear functional ℓ , the induced action on sections is $(g\ell)(v) = \ell(g^{-1}v)$, precisely the dual representation. The quark doublet factor would therefore contain $\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8}$ under color, rather than three copies of $\mathbf{3}$. The remaining multiplets also change. The central action on the lifted line bundle must likewise be tracked even when its action on the projective base is trivial. $\square$

A viable family construction needs a separate gauge-neutral multiplicity space F with G_{src} acting as identity, giving $F \otimes \mathcal{M}_+$. If $\dim F = 3$, the displayed anomaly sums are multiplied by three and remain zero. The physical selection of that space, its positive line bundle and operator, its decoupling from color, and its flavor interactions remain to be derived. The number three in a source carrier, a spatial coframe, and a family index cannot be counted as one theorem without explicit maps proving their different roles.

24.10 Spectral masses are compatibility relations, not selected constants

If an independent compact internal operator has eigenvalues λ_n and separation of variables produces a spacetime wave equation with mass term, then in the convention that λ_n has inverse-length squared units one obtains $m_n = \hbar\sqrt{\lambda_n}/c$. This is a conditional dispersion relation. Neither a compact inner coordinate nor the phase theorem selects the operator, its radius, its boundary conditions, or its coupling to the retained Weyl fields. For example rescaling that operator by a positive constant changes all such masses while leaving its polar phase and symmetry group unchanged. Consequently neither the internal parity gap, the auxiliary completion gap, nor the seven-cycle chord gap can be identified with an observed fermion or gauge-boson mass without an explicit matching calculation.

24.11 Selection tests, alternatives, and the next decisive calculation

The extension has sharp internal tests. Its retained module has rank sixteen, with degree multiplicities (1, 10, 5); dropping the neutral degree-zero sector produces rank fifteen but breaks the stipulated complete vacuum-connected module. The five charged multiplets still cancel the displayed anomalies if that singlet is removed, so anomaly cancellation does not require its physical existence. Keeping both parities as independent left-handed fields produces a mirror spectrum. Keeping only degree two produces ten components with nonvanishing color cubic anomaly $2 - 1 = 1$; keeping only degree four produces five with coefficient -1 . Multiplying by an arbitrary flavor space produces multiple generations and violates M1's no-multiplicity condition. These are discriminatory tests of M1–M4, not independent observations of SMM/DTT.

The next physical calculation should derive a local completion dynamics that selects M1–M4 before looking at the desired multiplets. It must explain the CAR interface, vacuum-connected even module, and spacetime Weyl interpretation together; preserve the anomaly-free measure; and determine whether the singlet remains physical, decouples, or acquires a mass. The finite H_{pair} is a reproducible selector witness, not a solution to those dynamical problems. Three

generations, flavor mixing, masses, coupling strengths, and Lorentzian geometry remain outside the matter extension. Autonomous finite-volume gauge dynamics has a separate explicit model in Section 2, but its coupling to a selected chiral continuum matter theory remains open. Reproducing one generation is a stronger compatibility result than recovering a frame group, but neither result establishes that nature realizes SMM/DTT.

Part III

Claim boundaries and proof obligations

25 What the combined reconstruction makes necessary

The strongest conclusion is a conditional structural theorem with a microscopic existence witness. It is stronger than a gauge analogy and weaker than an ontological uniqueness claim.

Theorem 25.1 (Combined conditional reconstruction). *The following implications hold, with their hypotheses kept separate.*

- (i) *A completion sufficient for the coherent future tests of Theorem 6.3 retains every transport distinction those tests resolve.*
- (ii) *Under R1–R3, faithful comparisons of independently unframed carriers obey the endpoint gauge law. R4 and the smooth transport hypotheses give the connection limit. For the stipulated 3 + 2 modules with parallel product-volume tensor, their frame group is $S(U(3) \times U(2))$.*
- (iii) *Invariant amplitude sewing and a neutral internal boundary condition select the corresponding Gauss sector. Frame twirling alone does not select that sector.*
- (iv) *For the specified independent singlet Hamiltonians with frozen links, the exact secular equation, gap, and single-trace Hessian of Theorems 2.1 and 2.3 hold. Their static two-block Hessian satisfies $\hat{\kappa}_1 = 12\hat{\kappa}_3 + 18\hat{\kappa}_2$.*
- (v) *For the bounded electric regularization and smallness assumptions of Theorem 2.6, the target Hamiltonian (14) approximates the dressed low sector with error (15) and time bound (16). A finite representation compression instead uses (19), including its boundary terms. The unbounded Hamiltonian has the separately stated self-adjointness and finite-volume heat-trace properties, not that global norm bound.*
- (vi) *The joint update of Section 20 simultaneously implements the specified completion and return test, preserving its calculated joint right algebra. Its reference preparations and link variables remain explicit resources. Under the independent assumptions M1–M4, the retained carrier yields the exterior one-generation module and displayed anomaly cancellations.*

Proof. Items (i)–(iii) are respectively Theorem 6.3, Theorems 6.5, 6.6 and 11.2, and Theorem 12.2 and Proposition 12.3. Item (iv) follows from the rank-one calculation and Corollary 2.4. The contour and direct-rotation proof, its finite-polynomial variant, and the separate domain argument prove (v). The explicit channel calculation in Section 20 proves the update claims in (vi), while Theorems 24.3 and 24.5 proves its conditional representation statement. $\square$

The behavioral theorem does not generate a 3 + 2 carrier; the carrier does not select a graph; sewing does not select a Hamiltonian; and an internal exterior module does not select Lorentzian chirality. The joint update reduces the number of disconnected implementation

steps by using the same return projector and hopping operator for records and their response. It does not remove the need to select its preparations, Hamiltonian, comparison variables, or spacetime interpretation. This is a compatible collection of proved conditional implications, not a uniqueness theorem from temporal ontology.

25.1 A counterfactual necessity test

Within a specified comparison class, necessity can be tested by removing one operation while keeping the physical targets and observations fixed.

Operation removed	Consequence under the stated hypotheses	What the test does not prove
All transport distinctions	Interference-distinguishable histories are merged; predictive sufficiency fails.	No preferred storage format is forced.
Additional relational comparison	Independent endpoint structural data give only a zero natural comparator.	A recorded shared reference remains an allowed implementation.
Completion outside a closed reversible algebra	Exact selector erasure cannot be implemented by a same-algebra reversible conjugation.	A larger unitary dilation is allowed.
Coherent internal sewing	Frame-invariant density matrices can retain nonsinglet sectors.	Gauss neutrality does not follow from twirling.
All virtual transfers in the singlet model	Its generated loop coefficient vanishes as t^4 .	Other Hamiltonians can generate loops by other mechanisms.
Flux control in the cyclic rotor family	The finite-region thermal trace diverges at $\kappa = 0$.	Neither a universal electric term nor its coefficient is selected.
Faithful spatial displacement data	The same local clock fits graph dimensions $q = 1, 2, 3, \dots$	Internal rank alone is not a dimension law.

These counterfactuals explain the proposed foundational roles of inner generation and completion. They do not exclude conventional gauge theory or other emergent constructions that supply the same structures under different names. To prove SMM/DTT necessary for a target class, the comparison class must be defined independently of these desired conclusions and all its alternative realizations must be addressed.

25.2 A unified set of quantitative tests

Test	Calculated result	Assumptions tested
Clock and defect response	$r_7 = [2 \sin(\pi/7)]^{-1}$	Regular source, declared cut, common response normalization.
Completion-control curvature	Rank twelve and the stated inverse-norm stability bound	Admissible control class A1–A4.
Frozen-link Hessian	$\hat{\kappa}_1 - 12\hat{\kappa}_3 - 18\hat{\kappa}_2 = 0$ exactly for the frozen-link Hessian	Two independent defining-carrier singlet loops.

Singlet-loop scale	Exact $\hat{\kappa}_m$ in (8); leading $2t_m^4/(m\Delta_m^3)$	Singlet return, unitary edges, specified penalty.
Rotor cycle hierarchy	$K_6/K_4 = (63/20)(\lambda/\Delta)^2$ at first cycle orders	Uniform zero- κ denominators and simple cycles.
Unclosed retained evolution	$I - T^*T = t^2CH(I - P)HC^*/\hbar^2 + O(t^3)$	Specified amplitude compression.
Dimension readout	Ball growth r^3 and return law $n^{-3/2}$ in the declared regular regime	Source-derived displacement and diffusion.
Matter content	Six multiplets, sixteen internal components, inherited charges and zero displayed anomalies	M1–M4.

Some tests are exact identities of their finite models; others are controlled leading coefficients or conditional large-scale laws. Their calibrations must be identified independently of the data they are used to explain. A designed simulation verifies a construction. Evidence for nature realizing SMM/DTT would require an independently selected physical implementation and a discriminating outcome shared by the proposed microscopic principles but not freely fitted by alternatives.

26 Scope and reproducibility

The transport construction alone does not derive matter or the dimension of physical space. Section 24 adds a conditional one-generation module and its inherited hypercharges; its spacetime Weyl interpretation is assumed. Section 16 proves that the local clock and record operations admit graphs of different dimensions. Lorentzian geometry, gravity, physical chirality selection, multiple generations, Higgs dynamics, Yukawa values, monopole particles, confinement, and a continuum mass gap remain open. They are not consequences of the frame stabilizer.

Finite-matrix checks concern the seven-state shift, reflection, Fourier ranks, the rank-five truncated shift, doubled defect kernel, and the standard-form fixed-algebra dimensions. They cannot validate the physical source axioms, smooth global descent, or continuum field dynamics. The accompanying executable verification package checks the displayed finite examples and scalings. It does not replace the proofs or certify continuum physics. The algebraic proofs and the local connection calculation are given explicitly; global source-compatible gluing remains a stated assumption.

27 Claim status and research priorities

Result	Scope and dependencies
Quaternionic source	Algebraic consequence of finite real associativity, norm composition, and noncommutativity.
Seven-state recurrence	Consequence of the stipulated minimal quadrature and regular-reset rules.
Complex $3 + 2$ carriers	Positive-frequency selection plus real doubled defects extended through the selected phase algebra.
Selector forgetting	Exact result for the specified left depolarization and block dephasing.

Raw cut-phase obstruction	Exact noncommutation, repaired by a declared scalar-extension completion functor.
No preferred right polarization	Requires absence of an additional retained right tensor; not absence of right observables.
Determinant-linked group	Stabilizer conditional on retained splitting and global total tensor.
Necessary gauge transport	Follows from R1–R3 and the structural-data-only comparator obstruction; R4 gives its connection limit.
Inner generation and completed record	Exact selector erasure is at $2 \rightarrow 2$ distance at least one from every reversible conjugation on the same algebra; this does not single out DTT or a particular channel.
Joint reference frames	The explicit joint update preserves shared right references. Their absence is a preparation restriction; exact preservation and erasure cannot be combined.
Spatial dimensionality	$\mathbb{Z}^q$ countermodels locate the gap. A displacement-growth criterion, regular diffusion calculation, completion cone, and dual point-source dimension constraint are conditional results; selection of their inputs remains open.
Kato curvature	Exact geometry of a smooth isolated spectral bundle; compatibility with the total tensor is additional.
Source accessibility	Bare oriented-frame variation is flat; declared open-completion controls access twelve origin curvature directions with a stability bound.
Full holonomy	Possible for suitable connections; pointwise accessibility is not a proof of global physical realization.
Curvature comparison cost	Polar comparison gives a smooth F^2 response; the specified singlet-return Hamiltonian independently generates its two-block trace form at fourth order and an exact all-orders static Hessian constraint.
Autonomous gauge dynamics	The saturated-electric model has an explicit sixth-order norm and dressed dynamics bound. Finite compression requires boundary polynomials. Electric parameters, microscopic selection, uncapped tail bounds, and continuum chiral dynamics remain open.
Invariant sewing	Coherent Haar sewing gives the neutral Gauss sector; twirling of density matrices does not.
Rotor normalization	Unpenalized integer circulations obstruct finite thermal trace; positive quadratic flux control suffices in the declared family without fixing its value.
r_7 response	Exact common-clock finite-source ratio; no particle-mass identification.
Response diagnostics	Primitive stiffness closure, independent central-cost residual, finite-record laws, and leakage bound.
One-generation module	Even exterior completion gives the six multiplets, inherited charges, and displayed anomaly cancellations under M1–M4.
Standard Model	Physical chirality selection, generations, couplings, masses, symmetry-breaking dynamics, and phenomenological matching remain open.

The next priorities are physical selection of the explicit joint update and its preparations; source-derived adjacency and dimension; a coframe and Lorentzian limit; weighted electric-tail and volume-uniform estimates; and physical CAR/chirality selection. Global tensor descent and the absence of additional structural references need independent justification. Necessity beyond this model requires a comparison class defined without building the desired conclusions into it.

28 Proof obligations and open problems

The finite-model proofs leave the following distinct obligations. Meeting one does not remove the hypotheses of another.

28.1 Selection of the source, record, and Hamiltonian interfaces

The source axioms in Section 7 must be motivated independently of the desired carrier ranks and stabilizer. In particular, one must justify associative norm composition, minimal quadrature, regular resetting, the scalar-extension cut interface, and retention of the product-volume tensor rather than an alternative tensor or reference preparation. A global tensor requires compatible patching, not only vanishing local trace curvature. The joint update in Section 20 makes reference consumption and surviving right correlations explicit; it does not derive the absence of a shared right reference.

An autonomous completion dynamics must also select the defining-carrier singlet, binding projector, auxiliary penalty, hopping, link configuration space, and comparison graph used in Section 2. The exact resolvent and Hessian prove what this Hamiltonian does. They do not prove why a common temporal source chooses it, or why other gauge-invariant microscopic Hamiltonians should yield the same response. A useful robustness analysis must identify which alternative interactions preserve the Hessian constraint and which generate an independent central term.

28.2 The four dimensional criteria

The countermodels in Proposition 16.1 require a microscopic dimensional reconstruction to meet all four tests from Section 16:

- (D1) Derive the completed-event graph, influence or comparison relation, adjacency, and edge weights from the update itself. Prove a stable large-scale graph class without inserting a cubic lattice.
- (D2) For finite graphs of scale L with $\Theta(L^3)$ vertices, prove $|B_x(r)| \asymp r^3$ for $1 \ll r \ll L$ and, for a specified unbiased diffusion, $p_{2n}(x, x) \asymp n^{-3/2}$ for $1 \ll n \ll L^2$, uniformly away from boundaries. Volume growth alone does not control diffusion on graphs with bottlenecks.
- (D3) Derive an operational displacement/coframe map into the reflection-odd source module $W_s \simeq \text{Im } \mathbb{H}$. Prove a rank-three continuum limit, a nondegenerate induced metric, and identification of source $SO(3)$ with spatial tangent-frame rotations. The same construction must preserve independence of spatial source frames and retained internal right frames, rather than impose a mixed tensor that reduces the claimed gauge stabilizer.
- (D4) Derive chronological order, a local causal propagation rule, and a four-dimensional Lorentzian continuum with three-dimensional spatial slices. An attached clock index or a conditional speed bound does not establish this limit.

The displacement-growth criterion of Proposition 16.2 and the point-source duality constraint of Theorem 16.3 address different parts of this problem. Their inputs must be obtained from the same microscopic rule; neither substitutes for D1–D4.

28.3 Electric tails, useful time estimates, and regulator removal

For the unbounded electric Hamiltonian, an auxiliary gap does not isolate all retained electric states from auxiliary excitations. Extending Theorem 2.6 therefore requires the weighted high-representation tail estimate in (22), together with compatible auxiliary dressing estimates. These estimates must be uniform in the specified initial-state window, cutoff, volume, and physical time range. A bare low-representation window is not invariant under hopping. The boundary polynomials in Section 2.3.1 must be retained until their contribution is controlled.

The explicit sixth-order bound establishes asymptotic control at fixed volume and regulator. Obtaining a useful numerical bound over magnetic evolution times at larger allowed hopping is a separate quantitative problem; agreement of finite-matrix examples does not improve the proven norm bound. Removing the electric cap, taking a thermodynamic limit, and taking a continuum limit are separate steps. None follows from the frozen-link resolvent or the finite-volume spectral gap.

28.4 Matter selection and chiral field theory

A physical completion mechanism must derive, rather than restate, M1–M4: the CAR interface with no independent multiplicity space, the full vacuum-connected even module, its spacetime Weyl interpretation, and the unmodified exterior gauge action. The finite internal parity penalty implements M2 algebraically; it is not a local Lorentz-invariant mass mechanism for removing mirror Weyl fields. The quaternionic $SU(2) \rightarrow SO(3)$ lift in Section 24.8 is a geometric interface and does not derive spin-1/2, CAR, or the spin–statistics theorem.

The displayed anomaly cancellations are consistency checks for the specified multiplets. Constructing the chiral quantum measure, proving physical mirror decoupling, coupling the fields to autonomous links, and controlling the interacting continuum theory remain open. Generation multiplicity, flavor mixing, fermion masses, Higgs dynamics, and Yukawa parameters are not selected by the internal exterior construction.

28.5 Continuum matching and physical discrimination

The exact frozen-link relation $\hat{\kappa}_1 = 12\hat{\kappa}_3 + 18\hat{\kappa}_2$ does not determine the electric coefficients $\beta_3, \beta_2, \beta_1$. A physical continuum limit requires additional matching of electric and magnetic terms, common limiting speeds, and control of auxiliary decoupling, renormalization, and residual completion responses. No prediction of observed gauge couplings follows from the static relation alone.

Finally, a physical test must identify the implementation and calibrate its parameters independently of the outcome being explained. The calculated finite-model identities can test a realization of the declared construction. Evidence for nature realizing SMM/DTT would require a discriminating consequence of independently justified microscopic principles, rather than agreement obtained by freely selecting their interfaces.

29 Conclusion

The most substantial new result is an exact solution of the static singlet-loop elimination. The physical resolvent branch is unique at all positive gaps and real hoppings, and the rank-two and rank-three auxiliaries have a sharp uniform gap bound. Their exact small-curvature Hessian contains no independent trace-square term. Consequently $\hat{\kappa}_1 = 12\hat{\kappa}_3 + 18\hat{\kappa}_2$ survives every hopping order of this frozen-link model, and equal block parameters give the exact ratio 3/2. Higher static orders change the coefficient and the full potential, but do not by themselves violate that identity. Electric dressing and new interactions lie outside this protection.

A separate finite-volume theorem restores electric motion with a specified bounded dispersion and proves a sixth-order norm remainder, an explicit dressed time-evolution bound, and its volume and regulator dependence. The representation-cutoff version includes folded and boundary terms rather than treating compressed links as unitary. The unsaturated electric model is a well-defined autonomous Hamiltonian; extending the norm estimate to it requires the stated high-representation tail control. No thermodynamic or continuum limit is inferred from a finite-matrix check.

The joint update uses coherent mobile-carrier evolution and the same singlet return projector for an operational record and an energetic completion penalty. Its fixed algebra, retained correlations, and short-time return statistics are calculated. It preserves shared right references when they are present. Product-volume coherence requires preparation and a phase resource; its preservation cannot be renamed spontaneous generation. This partial construction makes the microscopic interfaces explicit while retaining their physical costs.

These results sit within five distinct reconstruction steps: predictive retention, comparison covariance, internal-group selection, Gauss sewing, and gauge dynamics. The first two have general operational force; $S(U(3) \times U(2))$ additionally requires the constructed carrier modules and retained tensor. The polar-phase and exact-completion arguments, flat-source countermodels, and the rotor circulation obstruction delimit what the temporal foundations actually imply. The observable algebra alone does not faithfully encode the central phase action.

The update does not generate adjacency, a three-dimensional coframe, CAR composition, even-sector matter selection, or physical Weyl chirality. The dimensional criteria and point-source argument therefore remain conditional, as do M1–M4 and their familiar one-generation exterior representation. Spatial dimension, color, spin, and family multiplicity have separate mathematical roles. Selecting the physical update, deriving spacetime and chiral field dynamics, controlling limits, and matching parameters remain necessary before claiming a derivation of the Standard Model or empirical identification of SMM/DTT. The advance is a stronger set of proofs and quantitative model consequences, with the remaining assumptions exposed.

A Verification protocol and calculated examples

The accompanying `verify_gauge_v19.py` uses NumPy, SciPy, and Matplotlib. It writes `verification_results.json` and the figure used in this paper. The checks cover:

1. exterior-sector dimensions and charges, quotient descent, CAR identities, parity-selector gap, representation covariance, complete cubic-generator anomaly traces, Yukawa selection rules, and charge ambiguity tests;
2. the structural-data-only comparator constraint, spanning-tree link reduction, bare source-frame flatness, source-preserving curvature-control rank, primitive loop-test probability, and stiffness closure/error bounds;
3. the seventh power of the regular shift, real cut rank five, real doubled kernel rank two, and unit singular gap;
4. the twelve-dimensional gauge algebra, six-element covering kernel, and the exact cut-phase obstruction $\sqrt{6/7}$ and the scalar-extension realization $D_R \otimes I_2$, whose real kernel rank is four and complex kernel rank is two;
5. completely positive finite record channels, fixed-algebra dimensions, and phase-register coherence factors;
6. polar-overlap covariance under independent random frames;
7. convergence of graph plaquette trace deficits to $\|F_{xy}\|^2/2$ after division by a^4 ;

8. numerical frame-derivative curvature and projection-loss checks, including the leakage bound;
9. exact latitude phase and survival formulas, their asymptotic coefficients, and the central Wilson ratio.

The additional `verify_foundations_v19.py` checks exact cycle denominator sums, the non-Abelian singlet-loop coefficients and gauge covariance, perturbative remainder scaling, phase balance, retained evolution leakage, and the distinction between twirling and singlet projection. These checks verify finite formulas and provide reproducible regression evidence. Infinite-volume, global-topological, normalizability, and dimension claims rely on the hypotheses and analytic arguments stated in the text; finite matrices cannot certify their physical realization. The scripts are not a proof of physical source selection, global field realizability, or continuum quantum dynamics. Random-frame checks use a fixed seed for reproducibility. Finite-difference derivative checks use explicit small-step tolerances and do not replace symbolic identities.

A.1 Reproducibility checklist and coverage boundaries

The six supplied scripts perform finite-matrix, exact rational, and representation checks. They do not prove the quantified theorems, select the source axioms, determine measured gauge couplings, or establish a continuum limit. Every theorem, proposition, lemma, and corollary below retains its analytic proof in the manuscript. “Example check” means an independently calculated instance or witness; “analytic only” means that no dedicated executable check is supplied. A passing numerical example is not a verification of all hypotheses or all conclusions of a theorem.

Run `python3 run_verification.py` from the package directory. The runner stops on any failed assertion, unsuccessful process, or missing positive suite result; it writes `verification_summary.json`. All six suites passed for Python 3.12.14, NumPy 2.3.5, SciPy 1.17.0, and Matplotlib 3.10.8. The script suffix `v19` is retained to identify the original verification implementation; the revised exposition does not silently rename or alter the mathematics checked by those scripts. Numerical roundoff residuals and finite-difference values may vary slightly with the numerical environment.

Code	Script	Calculated output
G	<code>verify_gauge_v19.py</code>	<code>verification_results.json</code>
M	<code>verify_matter_v19.py</code>	<code>matter_verification_results.json</code>
F	<code>verify_foundations_v19.py</code>	<code>foundations_verification_results.json</code>
S	<code>verify_static_v19.py</code>	<code>static_verification_results.json</code>
D	<code>verify_dynamics_v19.py</code>	<code>dynamics_verification_results.json</code>
J	<code>verify_joint_v19.py</code>	<code>joint_verification_results.json</code>

Result-by-result coverage. The references in the following table use source labels, so that the checklist remains valid under section reordering. Multiple references in one row share the stated coverage; a partial check is explicitly distinguished from the analytic conclusion.

Table 13: Numerical coverage of labeled results; scripts test the stated instances and do not replace analytical proofs.

Result	Code	Executable coverage and boundary
No-go theorems 8.8, 10.1, 11.5 and 22.4	—	Analytic only: source-class alternative, selector erasure, global determinant obstruction, and absence of a generic infrared mass-ratio claim.

Result	Code	Executable coverage and boundary
Theorem 4.1	G, F	Component examples: seven-cycle, defect ranks, selector algebra, balanced tensor; not the complete conditional implication.
Propositions 5.1, 5.2 and 5.4 and Theorem 5.3	—	Analytic only: quotient sufficiency, descended inverses, polar factorization, and scale freedom.
Theorem 5.5	F	Example check of the closure-deficit identity and its short-time leading coefficient; the iff statement is analytic.
Theorem 6.1	F	Normalized erasure witness under one reversible conjugation; exact noninvertibility and the general obstruction are analytic.
Proposition 6.2	J	Full right matrix-unit basis and a retained Bell-state correlation in the two-record example; the multiplicative-domain argument is analytic.
Theorem 6.3	F	Distinguishable interfering-phase witness at $\theta = 0.7$; the universal statement is analytic.
Corollary 6.4	—	Analytic only: smooth transport reconstruction requires all stated regularity assumptions.
Theorems 6.5 and 6.6	G	Endpoint-center constraint has rank 13 and zero solution dimension; finite relational-link/tree witness. Minimality and smooth reconstruction are analytic.
Proposition 6.7	G	Four-vertex, five-edge graph: tree links become identity and the four-edge cycle transforms by root conjugation.
Proposition 6.8	G	Constant-projector/flat source-frame example; absence of an implication to interacting dynamics is an analytic counterexample argument.
Lemmas 8.1 and 8.3 and Corollary 8.2	—	Analytic only: source classification, intrinsic orientation, and the quadrature lower bound.
Lemma 8.5 and Theorem 8.6	G	Chosen seven-state shift and source-preserving embedded orbit; classification and minimality under the axioms are analytic.
Proposition 8.7 and Lemma 9.1	—	Analytic only: exclusion of other orders and the retained isotropic form.
Lemma 9.2	G	Fourier ranks of the explicit seven-cycle; uniqueness of the polar complex structure is analytic.
Proposition 9.3 and Theorem 9.5	G	Exact cut-commutator norm, rank-five cut, defect singular values, and scalar-extended kernel ranks.
Theorem 9.6	G	Exact-source singular/chord data; perturbation estimates and quantified spectral protection are analytic.
Theorem 10.3 and Corollary 10.4	G	Choi positivity, idempotence, fixed ranks and carrier preservation for $m = 2, 3$; the exact semigroup decay rate is analytic.
Theorem 10.6	G, J	Semigroup attenuation and fixed dimension in G; compatible discrete pinching and preserved product coherence in J. General complete positivity is analytic.
Proposition 10.7	M, J	Occupancy charges and compensated charge-angle rescaling; primitive integer uniqueness is analytic.
Proposition 10.8 and Lemma 11.1	—	Analytic only: independent group-action factorization and marker-fixed phase convention.
Theorem 11.2	G, J	Product determinant correction and a covariant reference transfer; stabilizer identification and global retention remain analytic/assumed.

Result	Code	Executable coverage and boundary
Corollary 11.3	G, M	Six covering-kernel elements and the reconstructed multiplets' descent congruence.
Proposition 11.4 and Theorem 12.2	—	Analytic only: global cocycle construction and Haar amplitude projector/Gauss equivalence.
Proposition 12.3	F	A normalized twirled density matrix has no invariant amplitude subspace in the charges $+1, -1$ example.
Corollary 13.2 and Proposition 13.4	—	Analytic only: differential transformation law and the affine space of compatible connections.
Propositions 13.3 and 14.1	G	Origin graph curvature and link covariance in an explicit rank-two example; general bundle formulas are analytic.
Proposition 15.1	G	Positive projector, defect connection, and curvature for an explicitly varying source frame.
Theorem 15.2 and Corollary 15.3	G	All twelve normalized generator controls; rank and singular-value bound for a perturbation of norm 0.03.
Propositions 16.1 and 16.2 and Theorem 16.3	—	Analytic only: different graph dimensions, the sufficient growth criterion, and electric/magnetic source dimension counting.
Proposition 17.1	—	Analytic only: the polar-factor minimization theorem; computing a polar link is not a test of uniqueness of its minimizer.
Theorem 17.2	G	Random independent endpoint frames, corrected two-block links, and closed-cycle conjugation.
Proposition 17.3	G	Controlled-loop rejection probabilities for random $m = 3, 2$ holonomies.
Proposition 17.4	G	Four shrinking plaquettes in a graph chart; the continuum/Riemann-sum assertion is analytic.
Theorem 17.5 and Propositions 17.6 and 17.7	G	A closed rate/penalty change, added central invariant, and a saturating norm-error example; universal classification/bound is analytic.
Theorem 18.1	G	Explicit graph curvature, two leakage coefficients, and shrinking-overlap loss; the general inequality is analytic.
Propositions 19.1 and 19.2	G	Independent products of frame overlaps/projection norms, asymptotic coefficients, and two central-phase checks.
Proposition 3.1, Theorem 3.3, and Corollary 3.4	—	Analytic only: exact constraint reduction, circulation symmetry, and thermal-trace statements; no infinite rotor is numerically truncated to claim these results.
Theorem 3.2	F	Exact rational sum over every ordering for cycle lengths 3, 4, 5, 6, 7; the all-length identity is analytic.
Theorem 2.1 and Corollary 2.2	S	108 Hamiltonian cases: independently assembled matrices versus spectral/rational resolvents, exact gaps, and identity energy comparison; uniqueness, strictness and the uniform gap are analytic.
Theorem 2.3 and Corollary 2.4	S	Five Hessian parameter sets, central/traceless/general generators, mixed-angle derivative and four two-block parameter sets; the absence of a trace-square term is proved analytically.
Theorem 2.5	S, F	Direct fourth-order coefficient checks; six weak-hopping remainder samples; unequal edges/penalties; the uniform remainder bound is analytic.

Result	Code	Executable coverage and boundary
Theorem 2.6	D	Canonical low-band Hamiltonians, evenness, gap bounds, sixth-order errors and Duhamel checks for one or two disjoint $U(1)$ squares; not a numerical non-Abelian continuum test.
Theorem 20.1 and Proposition 20.2	J	Superoperator fixed dimensions for $m = 2$ and constructed dark matrix units for $m = 3$; identity and generic holonomies. General fixed-algebra decomposition is analytic.
Proposition 20.3	J	Covariance, coherent reference transfer, product coherence, and failure of symmetric reduced preparation.
Theorem 20.4	J	Three holonomies, exact second/fourth moments, coherent returns at three times, and explicit remainder inequalities.
Proposition 21.1	—	Analytic only, conditional on the listed spacetime and derivative-expansion assumptions.
Theorem 22.2	G	The exact-source chord and cut-gap ratios; no matching to physical particle or coupling data.
Theorem 22.3	—	Analytic only: rigidity within the stipulated response class, not a general renormalization result.
Propositions 24.1 and 24.2 and Theorem 24.3	M	Explicit five-mode CAR matrices, pair closure, even-degree dimensions, occupancy charges, sector multiplicities, exterior group composition and derivative.
Proposition 24.4	M	Complement subsets exchange parity and reverse central charge; the full determinant/metric identification is analytic.
Theorem 24.5	M	Symmetric cubic and mixed gravitational traces for all 12 generators, four weak doublets, and rational mixed-charge examples; the Witten criterion and spacetime interpretation are analytic.
Proposition 24.6	M	Four Yukawa charge sums and the sole central-charge-compatible symmetric mass entry; representation invariance and coefficient freedom are analytic.
Unbroken-phase mass obstruction (corollary following Proposition 24.6)	M	Central-charge enumeration rules out charged bilinears in this single internal module; the gauge/Weyl mass statement is analytic.
Proposition 24.8	—	Analytic only: the color-dual action prevents reinterpretation as neutral family multiplicity.
Theorem 25.1	G–J	Only the component checks listed above; the combined conditional implication is analytic and retains every stated hypothesis.

Declared seeds and parameters. No suite uses experimental input or network retrieval. G uses seed 20261006; F uses seed 20261007; S uses seed 190071; M uses seed 1606. D and J are deterministic. The following settings identify each principal checked construction; tolerances and the complete executable matrices are specified in the scripts.

A. G (source, records, comparisons). The clock has $N = 7$ and marked basis vector e_0 . Selector ranks are $m = 2, 3$; the phase-register charges are $(0, -6, 6, 0)$ with $\gamma_\phi t = 0.07$. The rank-two graph is $Z = xI_2 + y i \text{diag}(1, -1)/2$, with central differences $h = 10^{-6}$, plaquette spacings $0.1, 0.05, 0.025, 0.0125$, and loss spacing 0.001 . Covariance compares $(x, y) =$

(0.03, 0.02) and (0.05, -0.01). The operational graph has edges (0, 1), (1, 2), (2, 3), (3, 0), (0, 2) and the first three as a spanning tree. Flat-source derivatives are at (0.2, 0.1) with step 10^{-6} . The source-preserving embedding uses $Z = 0.07I_3 + 0.03i \text{diag}(1, -1, 0)$ and complementary eigenvalue 7; accessibility uses all 12 Hilbert–Schmidt-normalized generators and a random perturbation rescaled to operator norm 0.03. Primitive coefficients are $(\kappa_3, \kappa_2) = (2, 3)$, $\chi = 0.02$, and $\eta = 0.01$; the closed changed pair is (2.8, 1.9). Latitude data use $b = 1/4$ and $N = 8, 16, 32, 64, 128, 256, 512, 1024$; central checks use $\Phi = 0.001, \pi/2$.

- B. M (conditional matter).** Five modes give the 32-dimensional CAR module, the 16 even subsets, and central charges $(-2, -2, -2, 3, 3)$. Exterior derivatives use $\delta\theta = 10^{-6}$, the parity selector gap is 2, and cubic traces use all triples with replacement from 12 generators. The five anomaly-compatible charge examples use $(x, h) = (-2, 7), (0, 1), (3, 9), (1, 0), (1, 3)$. The supplied code also checks exact rational mixed $Y-(B-L)$ coefficients and four Yukawa charge sums.
- C. F (finite witnesses and cycle sums).** Every edge ordering is enumerated using exact fractions for lengths 3 through 7. Singlet checks use ranks 2, 3, gap 1, random links and random local frames, with hopping 1 for coefficient extraction, 0.07 for covariance, and 0.08, 0.04, 0.02 for direct-energy convergence. The balanced tensor uses real dimensions 6 and 4. The closure witness uses a Hermitian 5×5 matrix, its first two coordinates, time 0.013, and short times 0.02, 0.01, 0.005 with $\hbar = 1$. Twirling uses charges $+1, -1$ and phase 0.73; the erasure witness has $m = 2$; history interference uses $\theta = 0.7$.
- D. S (exact static model).** The 108 resolvent/gap cases use $m = 2, 3$, $\Delta = 0.6, 1, 2, 3$, $t = 0, 0.035, 0.22, 0.7, 2, -0.7$, and identity, random, or central- π holonomy. Hessians use $(m, t, \Delta) = (2, 0.11, 1), (3, 0.11, 1), (2, 0.6, 0.7), (3, 0.6, 0.7), (3, 1.7, 0.9)$; the five-point step is 0.015, mixed eigenangle step 0.03, and unit-change factor 3.7. Two-block tuples $(t_3, \Delta_3, t_2, \Delta_2)$ are $(0.12, 1, 0.12, 1), (0.65, 0.7, 0.65, 0.7), (0.17, 1.3, 0.31, 0.8)$, and $(1.2, 0.7, 0.24, 1.8)$, with central step 0.003. Unequal-edge checks use ratios $(0.6, 1.1, 0.8, 1.2)$, position gaps $(0.7, 1.3, 2.1)$, root nonsinglet gap 1.4, $m = 2, 3$, and $\varepsilon = 0.08, 0.04, 0.02$. Weak hopping uses $m = 3$, $\Delta = 1$, eigenangles $(-0.8, 0.25, 1.1)$ and $t = 0.09, 0.07, 0.05, 0.035, 0.025, 0.018$. Small-curvature remainders use the same diagonal generator, $t = 0.6$, $\Delta = 1$, and $h = 0.16, 0.08, 0.04, 0.02$.
- E. D (finite-volume dynamics).** Set $\Delta = \hbar = 1$, electric coefficient $\alpha = 0.07$, and (Peter–Weyl cutoff, disjoint-square count) $(L, N_\square) = (1, 1), (2, 1), (4, 1), (1, 2)$. The radius is $r = 1/(32N_\square)$ in these cases, and $\varepsilon/r = 0.85, 0.70, 0.55, 0.40$. Duhamel comparisons use time $T = 2$. Boundary-coefficient comparisons use $\alpha = 0$, $L = 1, 2, 3, 5$, and input window $|k| \leq K = 1$. The separate constants table uses $N_\square = 1, 2, 4, 8$ and $L = 1, 2, 5, 10$ without constructing the larger matrices.
- F. J (joint cycle and coherent return).** Fixed-algebra checks use $\Delta = 1$, $t = 0.17$; the $m = 2$ generic eigenphases are (0.37, 1.19) and $\tau/\hbar = 0.43$, while $m = 3$ uses (0.21, 0.83, 1.47) and explicitly constructed dark matrix units. Identity holonomy is also checked at each rank. Joint correlations use selector weights (0.1, 0.2, 0.3, 0.4), a two-qubit Bell state, and an active phase 0.7. Determinant pinching uses charges (0, -1, 1, 0), angle $2\pi/3$ (compensated by $2\pi/18$ after a sixfold charge rescaling), covariance phases $(a, b) = (0.53, -0.17)$, coherent reference phase 0.71, and symmetric-pair evolution parameter 0.4. Return checks use $\Delta = 1$, $t = 0.37$, eigenphases (0.6, 0.6), (0.37, 1.19) or (0.21, 0.83, 1.47), and $\tau/\hbar = 0.4, 0.2, 0.1$.

Selected numerical outcomes. The independently diagonalized static matrices agree with the exact energy, rational secular equation, and gap formula to maximum absolute residuals 7.87×10^{-15} , 1.33×10^{-14} , and 8.44×10^{-15} , respectively. The largest static Hessian error is

6.77×10^{-11} ; the largest inferred trace-square coefficient magnitude is 1.41×10^{-11} and the largest two-block central Hessian residual is 1.93×10^{-10} . These are finite-difference/roundoff residuals, not analytic error bounds for arbitrary parameters.

The measured log-log slope of the static weak-hopping remainder is 5.9779. In D, slopes of the canonical operator error are 5.9972, 5.9967, 5.9965, 5.9993 in the four declared cases, and the maximum actual operator-error/theorem-bound ratio is 8.82×10^{-9} . The maximum measured dynamical error at $T = 2$ is 1.06×10^{-8} . This large separation from the conservative proved bound does not justify extending the demonstrated time range to a magnetic evolution time, nor imply a regulator-uniform estimate. Global second- and fourth-order errors relative to the naive Wilson boundary formula are 1 and 2 for every tested cutoff, whereas the $|k| \leq 1$ restricted error vanishes here for $L \geq 2$. The general four-transfer padding condition remains a sufficient condition, not a claimed optimal threshold for this example.

J gives residual auxiliary fixed dimensions 9, 68 at rank two and 49, 418 at rank three for generic and identity holonomy respectively. The tested Bell-state right marginal and every right matrix unit are preserved; the active frame change of the joint state has norm 0.48493, illustrating that local unpolarized marginals do not erase a shared joint reference. The coherent-return remainder halving ratios lie between 63.2403 and 63.8102, consistent with the calculated sixth-order short-time behavior. M gives exact zero CAR/anomaly trace residuals in its explicit basis, exterior-composition and unitarity residuals below 1.26×10^{-15} , and the retained $1 + 10 + 5 = 16$ sector dimensions. F gives exact cycle sums 6, 20, 70, 252, 924 and the coefficients $5/2$, $63/8$, and ratio $63/20$. All are calculated model outputs.

Figure regeneration. G writes `figures/finite_record_scaling.pdf`; S writes `figures/exact_singlet_static.pdf`; D writes `figures/effective_dynamics_and_cutoff.pdf`. Each also writes a PNG. Their captions specify the plotted observables, parameter ranges, units and guide curves. No figure contains observed data, fitted physical couplings, or a continuum extrapolation.

A.2 Independent exact static checks

The script `verify_static_v19.py` constructs the original $4m^2$ auxiliary matrices from four links, before gauge fixing. It compares their lowest two eigenvalues with a separately evaluated free spectral resolvent and the rational secular expression. The 108 cases use $m = 2, 3$, $\Delta = 0.6, 1, 2.3$, and $t = 0, 0.035, 0.22, 0.7, 2, -0.7$, with identity, random-unitary, and central $-I$ holonomies. Thus the tests include parameters outside the original weak-hopping isolation interval. The largest ground-energy difference is 7.9×10^{-15} , the largest rational-secular residual is 1.4×10^{-14} , and the largest exact-gap difference is 8.5×10^{-15} . Numerical comparisons support the analytic all-parameter gap proof; they are not its justification.

Five-point second differences of independently diagonalized energies check traceless, central, and general Hermitian directions, including $t = 1.7$, $\Delta = 0.9$. Their largest absolute stiffness error is 6.8×10^{-11} . The inferred coefficient of $(\text{Tr } X)^2$ is below 1.5×10^{-11} ; independent sign flips of two eigenangles also check the vanishing mixed Hessian. Equal-parameter pairs give the exact ratio $3/2$. Unequal examples include $(t_3, \Delta_3; t_2, \Delta_2) = (0.17, 1.3; 0.31, 0.8)$ and $(1.2, 0.7; 0.24, 1.8)$, with ratios approximately 34.3603 and 0.0106483, respectively. Their direct central Hessians and the exact weighted relation agree within 2×10^{-10} . This large variation illustrates why the equal-parameter ratio requires its microscopic equality assumption.

The inhomogeneous check uses edge hopping ratios

$$(0.6, 1.1, 0.8, 1.2), \quad (\Delta_1, \Delta_2, \Delta_3) = (0.7, 1.3, 2.1),$$

with root-nonsinglet penalty 1.4. Original matrices and their reduced resolvents give the fourth-order loop coefficient in (12) within 10^{-15} for $m = 2, 3$. Direct holonomy-energy differences at

common hopping scales 0.08, 0.04, 0.02 have scaled-error halving ratios between 3.979 and 3.997, supporting the sixth-order remainder scaling in this unequal-parameter example.

t_3	Δ_3	t_2	Δ_2	$\hat{\kappa}_3$	$\hat{\kappa}_2$
0.12	1	0.12	1	0.000126768	0.000190152
0.65	0.7	0.65	0.7	0.023230631	0.034845947
0.17	1.3	0.31	0.8	0.000228660	0.007856822
1.2	0.7	0.24	1.8	0.048005667	0.000511179

Table 14: Exact static stiffnesses in the stated energy units. The first two rows have exact ratio 3/2; unequal block parameters need not.

At $m = 3$, $\Delta = 1$, and holonomy eigenangles $(-0.8, 0.25, 1.1)$, the fourth-order hopping approximation is checked at $t = 0.09, 0.07, 0.05, 0.035, 0.025, 0.018$. The observed log-log slope of its absolute remainder is 5.978, consistent with the proved sixth-order bound. A separate small-curvature calculation verifies a fourth-order remainder after the exact Hessian. These two expansions vary different parameters and are not conflated.

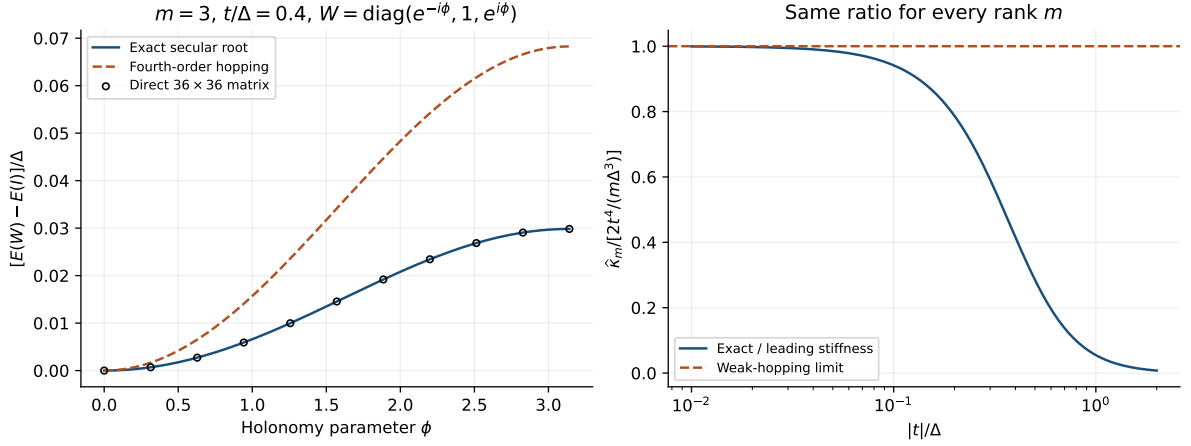


Figure 2: Calculated exact static singlet-loop outputs. Left: $m = 3$, $\Delta = 1$, $t = 0.4$, and $W = \text{diag}(e^{-i\varphi}, 1, e^{i\varphi})$. The ordinate is $[E(W) - E(I)]/\Delta$ at 151 equally spaced $\varphi \in [0, \pi]$; direct 36×36 matrix energies are marked at every fifteenth point. The dashed curve is the fourth-order hopping approximation and has no small-hopping error guarantee at this t/Δ . Right: exact stiffness divided by $2t^4/(m\Delta^3)$ at 180 logarithmically spaced $|t|/\Delta \in [0.01, 2]$, with $m = 3$ representing the rank-independent normalized ratio. The dashed line is its weak-hopping limit. These are calculated model outputs, not observed data.

A.3 Controlled finite dynamics, electric dressing, and cutoff boundary tests

The script `verify_dynamics_v19.py` tests the finite-matrix reduction in a fully specified $U(1)$ square sector. A mobile charge $+1$ at position j and a stationary charge -1 at the root satisfy Gauss law in the basis (k, j) with link fluxes $n_e = k + \mathbf{1}_{e < j}$, $e = 0, 1, 2, 3$. Forward hops with $j < 3$ leave k fixed; the hop $3 \rightarrow 0$ raises k . The true product Peter-Weyl cutoff $|n_e| \leq L$ is imposed on every link. Thus $j = 0$ admits $-L \leq k \leq L$, whereas $j > 0$ admits $-L \leq k \leq L - 1$. The full dimension is $8L + 1$ and retained dimension is $2L + 1$. Every omitted boundary hop is actually removed; no compressed link is treated as unitary.

In units $\Delta = \hbar = 1$, the matrices are $A = Q$, B equal to the negative hopping adjacency, and $D = 0.07 \sum_e n_e^2$. The electric term depends on the mobile position in this Gauss-reduced

representation, but commutes with the retained projector and position parity. Direct diagonalization followed by the normalized-overlap isometry gives the canonical low-band Hamiltonian. It is compared with the full second- and fourth-order compressed polynomial, including folded and boundary terms. For $L = 1, 2, 4$ on one square, and two disjoint $L = 1$ squares, the observed sixth-order slopes are respectively 5.9972, 5.9967, 5.9965, 5.9993. Each case uses $\varepsilon/r = 0.85, 0.70, 0.55, 0.40$ within the proved analytic radius. The full matrix dimensions are 9, 17, 33, 81 and their electric norms are 0.28, 1.12, 4.48, 0.56. At the largest admitted ε in each case, the directly measured low/high-cluster gaps are 0.96332, 0.96321, 0.96316, 0.98126, all exceeding the respective Weyl lower bounds $1 - 2(\varepsilon b + \varepsilon^4 d)$, which are approximately 0.89375. The gaps at the other tested parameters also satisfy that bound.

All calculated errors satisfy the explicit theorem bound; the largest ratio of observed error to that conservative bound is 8.9×10^{-9} . Dressed time evolution at $T = 2$ additionally satisfies the Duhamel estimate. This tests a bound proved analytically, not an optimality claim about its constant. The electric term is present, so its higher-order dressing is included in the measured remainder. The two-square example checks finite-volume accounting, but supplies no thermodynamic-uniform estimate.

At the cutoff boundary the naive Wilson target has a second-order coefficient error of norm one and a fourth-order coefficient error of norm two, independent of L in the displayed examples. The coefficient blocks on $|k| \leq 1$ agree after sufficient padding; in this particular square they already agree at $L = 2$, and the general four-transfer sufficient padding is also checked at $L = 5$. This distinction is why a gauge-preserving cutoff is not silently identified with the uncompressed model. The tests are for the stated Abelian representation sector; the non-Abelian theorem is supported by its proof and the separate static non-Abelian checks, not by pretending that this small matrix exhausts its representation content.

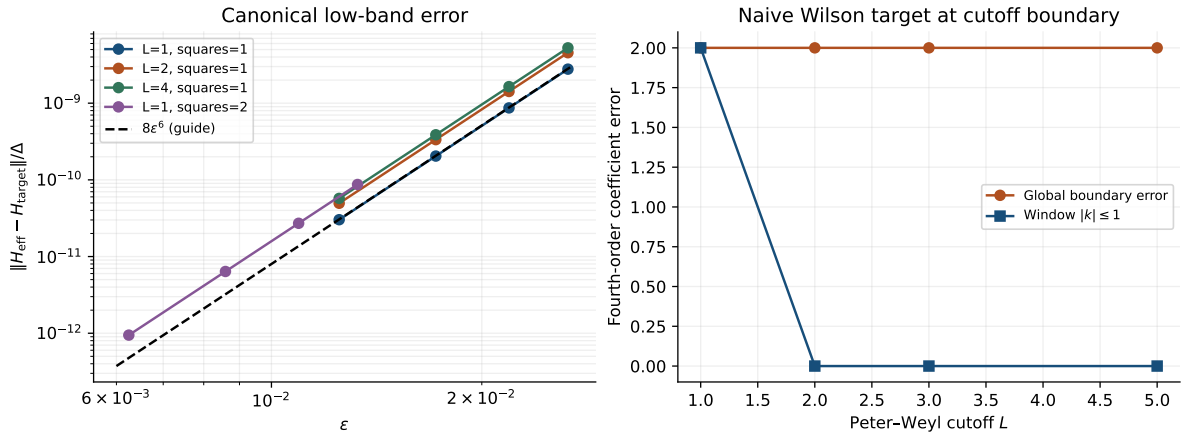


Figure 3: Calculated finite-matrix dynamics and cutoff outputs. Left: $\Delta = \hbar = 1$, electric coefficient $\alpha = 0.07$, and $(L, N_{\square}) = (1, 1), (2, 1), (4, 1), (1, 2)$ for one or two disjoint $U(1)$ squares. The ordinate is $\|H_{\text{eff}} - H_{\text{target}}\|/\Delta$ at $\varepsilon/r = 0.85, 0.70, 0.55, 0.40$, with $r = 1/(32N_{\square})$. The dashed $8\varepsilon^6$ line is only a scaling guide, not the theorem bound. Right: with $\alpha = 0$ and $L = 1, 2, 3, 5$, global and $|k| \leq 1$ -restricted fourth-order coefficient errors relative to the naive Wilson target. Agreement in this window for $L \geq 2$ does not sharpen the general sufficient four-transfer padding condition. These are calculated model outputs, not observed data.

A.4 Joint-update algebra and reference verification

The script `verify_joint_v19.py` checks the auxiliary cycle superoperator and its commutant independently for $m = 2$. Their fixed algebra dimensions are 9 for generic holonomy and 68 at identity. For $m = 3$ it constructs each residual matrix block and tests its commutators,

obtaining dimensions 49 and 418. These are dimensions of the auxiliary fixed algebra, not counts of physical particle states. The full tensor-product fixed algebra follows from the proved factorization.

A two-record $m = 2$ check preserves every joint right matrix unit and a Bell-state correlation $\langle X \otimes X \rangle = 1$, although both local marginals are maximally mixed. Thus locally unpolarized records can still contain a shared reference. The finite phase pinching has fixed dimension six, preserves the product coherence, and deletes the separate coherences. Rescaling its charge and sampling angle together is checked explicitly; rescaling the charge alone can alias the finite twirl to identity.

The covariant reservoir swap transfers a supplied determinant coherence of modulus $1/2$ into the product-volume register. Symmetric vacuum preparation with a covariant pair-creation pulse gives zero reduced coherence, as the preparation obstruction requires. For $\Delta = 1$, $t = 0.37$, and times $\tau/\hbar = 0.4, 0.2, 0.1$, direct spectral survival probabilities verify (79) for central and noncentral rank-two holonomies and a rank-three example. Absolute remainders decrease by approximately 64 on halving time, and all obey the analytic bound. A central rank-two holonomy with angle 0.6 has forced-circuit probability one but a nonzero coherent-waiting signal. These are implementation checks of the given update and preparations, not evidence that those inputs are uniquely selected in nature.

A.5 Matter verification values

The independent exterior verifier builds all 32 CAR basis states and retains 16 even states. Pair closure from degree zero reaches exactly those sixteen states. It checks all 364 unordered triples of twelve Hermitian Lie-algebra generators through $\text{Tr}[T_a\{T_b, T_c\}]$, obtaining zero for every displayed cubic anomaly tensor component. Generator traces also vanish. Random exterior-lift composition and unitarity residuals are below 2×10^{-15} ; a central-generator finite-difference residual is below 5×10^{-11} . Quotient congruences, complementary-parity conjugation, the gap of the finite selector, charge-allowed mass entries, four Yukawa sums, and five alternative anomaly-free charge assignments are checked. These numerical and exact finite checks do not validate M1–M4 as physical laws.

A.6 Hand-checkable values

For the seven-cycle,

$$r_7 = 1.152382435481243\dots, \quad 2\cos(\pi/7) = 1.801937735804838\dots$$

$$\|[P_\Omega, J]\| = 0.925820099772551\dots$$

For $b = 1/4$ the asymptotic coefficients are

$$N^2(\Phi_N - \pi/2) \longrightarrow -\pi^3/8, \quad N[-\log S_N] \longrightarrow 3\pi^2/4.$$

For a graph generator $T = i \text{diag}(1, -1)$ and $B = T/2$, $\|F_{xy}(0)\|^2 = 2$ and the plaquette trace deficit divided by a^4 approaches one. Here $q_x = 2$, $q_y = 1/2$, so the leakage bound gives $\|F\| \leq 2$, consistent with $\|F\| = \sqrt{2}$.

B Assumption sensitivity and alternative models

The source algebra is forced only inside the finite associative norm-composition class. General real operator algebras are not division algebras. Seven is selected only after one real quadrature pair per noncentral direction and an orthonormal regular orbit are required. The scalar-extension defect doublet is a mathematically coherent choice, but an alternative completion functor can have

N	Φ_N (rad)	$ \Phi_N - \pi/2 $	S_N
8	1.507836086	0.0629602	0.394243265
32	1.567002245	0.00379408	0.793413021
128	1.570559732	0.000236595	0.943809283
512	1.570781542	1.47851e-05	0.985646556
1024	1.570792631	3.69624e-06	0.992797347

Table 15: Exact finite-record values for the specified latitude carrier at $b = 1/4$. These are calculated model outputs, not observed data.

different retained rank. Standard-form separation is a chosen resolution of the selector-erasure obstruction; that obstruction does not prove its uniqueness.

Retaining two separate volume tensors would instead reduce the frame group to $SU(3) \times SU(2)$, while retaining neither would allow $U(3) \times U(2)$. Retaining a designated right doublet polarization reduces its stabilizer further. These alternatives make the determinant-linked construction sensitive to record content rather than solely to temporal counting. Flat projectors can realize all finite source relations without any nonzero curvature. A physical source family must support fluctuating transport with a calculated energy and temporal response if a force is claimed. It need not exclude flat configurations or a flat vacuum.

The polar-link cost chooses a particular gauge-invariant comparison functional. Other invariant costs introduce independent central stiffness or higher-derivative terms. The two-block central coefficient relation is therefore a model discriminator, not a universal identity. An eventual dynamical theory must show which fluctuations are independent, what measure or Hamiltonian governs them, and whether a Lorentzian continuation is justified.

C Data and code availability

All six original verification scripts, their runner, calculated JSON outputs, and three regenerated figures accompany this revision. The result-by-result checklist in Appendix A.1 records executable coverage, analytic-only obligations, seeds, and parameter grids. The original script suffix `v19` is retained for traceability; all six suites were rerun for the revised manuscript. Inputs are explicit matrices and dimensionless model parameters; no experimental dataset is used. The package includes the revised LaTeX, compiled PDF, source diff, change summary, and extraction map. Compile from its root directory so that the relative figure paths resolve.

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