

Temporal Completion, Gauge Transport, and Spectral Geometry: Toward a Conditional Unification Framework for the Duality of Time Theory

Mohamed Haj Yousef

Department of Physics, United Arab Emirates University
Al Ain, United Arab Emirates

3 October 2026

Abstract

Can the Duality of Time Theory unify gravity, electromagnetism, and the weak and strong interactions? A defensible answer requires distinguishing a common organizing language from a derivation of the physical theory. This paper develops a self-contained conditional framework in which generative histories, completed records, real orthogonal dynamics, internal transport, and spectral geometry are organized in a single dependency structure. Its proved mathematical results concern behavioral descent, positive-frequency phase selection, residual symmetry after phase selection, transport on completion networks, the obstruction to descending transport to records, and the local gauge and curvature laws of descended transport. Explicit counterexamples establish that these data alone do not select the Standard Model gauge algebra, particle representations, or numerical coupling constants. Under separately stated manifold, finite-geometry, heat-kernel, and physical-continuation hypotheses, an almost-commutative spectral action provides a common effective description of Einstein gravity, Yang–Mills fields, a Higgs sector, and fermions. We calculate one-generation anomaly cancellation, representation-trace normalization of gauge couplings, electroweak mixing, and the dimensional scale dependence of the gravitational term. The three published dual-time foundations are used within their established scope; no unpublished DTT manuscript is required. The outcome is a mathematical reconstruction program with explicit consistency tests, not an unconditional derivation of the Standard Model or a completed quantum theory of gravity. Its central research problem is to derive the internal algebra, chiral bimodule, dynamics, and spectral weights from a specified completion law rather than to prescribe them from observed particle physics.

Keywords: dual time; completion records; real commutant; phase selection; gauge transport; holonomy; spectral action; Standard Model; emergent gravity.

Contents

1	Introduction: what would count as unification?	3
2	Generative histories and completed records	4
3	Real dynamics, phase, and surviving internal symmetry	5
4	Transport on a completion network	7
5	When does microscopic transport descend to records?	8
6	Smooth gauge geometry from local record transport	9
7	What is required to select the Standard Model algebra?	11
8	Matter, chirality, hypercharge, and anomalies	12
9	Finite spectral transitions and the Higgs sector	14
10	Spectral geometry and a common action	15
11	Record geometry and the gravitational sector	18
12	Generalized transport: common curvature and its limits	20
13	Coupling constants, normalization, and scale matching	21
14	A conditional assembly theorem	24
15	Consistency of the quantum and continuum interpretation	26
16	A constructive research program	27
17	Discussion and conclusion	28
A	An explicit finite benchmark representation	28
B	Network action and its continuum expansion	30
C	A small model and its underdetermination	31
D	Notation and dimensional audit	32

1 Introduction: what would count as unification?

The four familiar interactions are described by two different kinds of effective geometry. Gravitation acts through spacetime geometry, whereas the electromagnetic, weak, and strong interactions act through connections on internal bundles over spacetime. A shared mathematical vocabulary does not by itself explain why these particular geometries, matter representations, and couplings occur. Any proposed unification must identify the data from which they follow, the rules that determine those data, and the predictions that remain after all inputs have been counted.

The Single Monad Model (SMM) and the Duality of Time Theory (DTT) suggest an explanatory ordering in which generative processes precede completed observable records. In this paper, this ordering is used as a research principle. It is not substituted for an operator algebra, an action, or a proof. The primitive distinction is between the parameter that organizes a reversible generative process and the time assigned to observable records. Neither a second observer clock nor a compact timelike spacetime coordinate is required.

Three published mathematical developments provide the starting point. The completion and real-commutant framework identifies compatible phase operators and treats cone readout conditionally [1]. The focused phase-selection construction identifies the polar phase of an oriented real orthogonal flow and states the zero-mode and composition limitations [2]. The dual-time topological framework studies temporal transport and projected memory dynamics [3]. Their established scope is the motivation for extending the architecture to transport between records; it does not include a demonstrated Standard Model or gravitational dynamics.

There is also an established spectral approach to effective geometric unification. Noncommutative geometry supplies a language in which a finite internal geometry and a continuous spin geometry are combined [8, 9]. The spectral action then yields gravitational and particle-physics terms under specified geometric input [10, 11, 12]. That construction is used here as a conditional downstream module. Its known results are attributed to that literature. DTT would add a substantive new contribution only if its primitive completion law selected, or tightly constrained, the required finite geometry and action.

1.1 Four levels of explanatory achievement

It is useful to distinguish four claims.

- (i) *Common description*: several interactions can be written in a shared bundle or spectral notation.
- (ii) *Common dynamics*: one specified functional supplies gravitational, gauge, scalar, and matter terms with correlated coefficients.
- (iii) *Structural derivation*: a primitive law selects the spacetime dimension, internal algebra, physical representations, and chirality without importing them as empirical assignments.
- (iv) *Predictive fundamental theory*: a consistent quantum construction yields nontrivial observables, with controlled limits and a demonstrable relation to experiments.

The present paper proves several components of level (i), explains a conditional route to level (ii), and formulates the missing tasks at levels (iii) and (iv). These levels should not be conflated. In particular, a direct sum of four connections is not evidence that four coupling constants have a common microscopic origin.

1.2 Scope and conventions

Natural units $\hbar = c = 1$ are used unless explicitly restored. The physical Lorentzian signature is $(-, +, +, +)$. The spectral heat-kernel calculation is Riemannian and positive definite; its physical continuation is an additional hypothesis. Gauge connections are anti-Hermitian. Thus

$$\nabla = d + \mathcal{B}, \quad \mathcal{F} = d\mathcal{B} + \mathcal{B} \wedge \mathcal{B}. \quad (1.1)$$

With Hermitian generators T_a , $\mathcal{B}_\mu = -igA_\mu^a T_a$. The real-linear phase J satisfying $J^2 = -\text{Id}$ is distinguished throughout from the antiunitary real structure $\mathcal{J}$ of a spectral triple. A frequency operator, an inverse-length Dirac operator, and a mass operator are also kept distinct.

1.3 Claim status and contribution

Statements labeled theorem or proposition follow from the hypotheses printed with them. A hypothesis is input, not a result. A reconstruction identifies a familiar object in supplied data; it is not a derivation of that input from SMM ontology. The principal contribution of this paper is the formulation of an explicit completion-to-transport interface, including its descent obstruction, and the consequences of separating selected complex phase from internal multiplicity symmetry. The elementary proofs are included so that none depends on an unpublished source. Novelty of the full assembly relative to the extensive literature remains a question for scholarly review.

2 Generative histories and completed records

2.1 Behavioral equivalence

Let X be a set of generative states, $\mathbf{H}$ a monoid of histories acting on X , and $Q : X \rightarrow \mathbf{R}$ a completion map into records. Let $\mathcal{O}$ be a family of tests on records. The identity belongs to $\mathbf{H}$, and multiplication is chosen so that $(hk)x = h(kx)$.

Definition 2.1 (Completed-future equivalence). For $x, y \in X$, define

$$x \sim y \iff o(Q(hx)) = o(Q(hy)) \text{ for every } h \in \mathbf{H}, o \in \mathcal{O}. \quad (2.1)$$

The observable state space for these tests is $X/\sim$.

Proposition 2.2 (Autonomous descent). *The relation (2.1) is an equivalence relation and is preserved by every history. Therefore $[x] \mapsto [hx]$ defines a monoid action on $X/\sim$.*

Proof. For each (h, o) , equality of the values of oQh defines an equivalence relation; their intersection is again one. If $x \sim y$, then for any $k, h \in \mathbf{H}$ the equality of tests for hk implies $oQ(h(kx)) = oQ(h(ky))$. Hence $kx \sim ky$. The induced action is well-defined, and its multiplication and identity laws follow from the original action. $\square$

The set of histories used in the tests matters. If only a subclass is used, stability under right concatenation must be imposed to obtain the same conclusion. Nor does an instantaneous equality $Q(x) = Q(y)$ guarantee future equality; the future tests are the explicit autonomy condition.

Remark 2.3 (Reversibility of a genuine group quotient). If $\mathbf{H}$ is a group and the equivalence relation is preserved by every group element, the descended action is also a group action and is reversible. Irreversible record dynamics requires restricting to a forward semigroup, enlarging the reduced state description, or admitting memory. Information loss in a completion map alone does not turn an exactly descended group action into a noninvertible one.

2.2 Hilbertization is additional data

An observable quotient is a set, not a Hilbert space. A positive state on a record $\ast$ -algebra can yield a GNS construction; alternatively, a measure and a space of square-integrable record functions can be supplied. Either choice requires positivity, a completion norm, and a representation. These inputs cannot be concealed in the phrase “one Monad generates all states.”

The same distinction applies to tick discreteness. A discrete shift gives a representation of $\mathbb{Z}$. A strongly continuous orthogonal flow gives a representation of $\mathbb{R}$. A logarithm or continuum limit that connects these descriptions is extra structure. For a discrete unitary shift, logarithms need not be unique; selecting an energy branch is a physical assumption. Accordingly, the next section explicitly assumes a continuous orthogonal flow instead of claiming it follows from counting ticks.

2.3 Projected memory without a second Lorentzian time

An elementary exact elimination illustrates the distinction between a generative state and a record. Let $r(t)$ and $z(t)$ obey bounded linear evolution

$$\dot{r} = Ar + Bz, \quad \dot{z} = Cr + Ez. \quad (2.2)$$

Variation of constants gives

$$z(t) = e^{tE}z(0) + \int_0^t e^{(t-s)E}Cr(s)ds, \quad (2.3)$$

$$\dot{r}(t) = Ar(t) + Be^{tE}z(0) + \int_0^t Be^{(t-s)E}Cr(s)ds. \quad (2.4)$$

The reduced record can have memory even when the full evolution is local in t . This standard projection mechanism [15, 16] is relevant to dual-time modeling but does not require an extra metric time dimension. It also shows why a chosen instantaneous projection need not satisfy autonomous descent. The memory kernel and initial hidden state carry the omitted information.

3 Real dynamics, phase, and surviving internal symmetry

3.1 The real commutant

Let $U : G \rightarrow O(\mathcal{H}_{\mathbb{R}})$ be an orthogonal representation. Its real commutant is

$$\mathfrak{C}_U = \{T \in \mathcal{B}_{\mathbb{R}}(\mathcal{H}_{\mathbb{R}}) : TU(g) = U(g)T \ \forall g\}. \quad (3.1)$$

A compatible orthogonal complex structure is an element

$$J \in \mathfrak{C}_U, \quad J^* = -J, \quad J^2 = -\text{Id}. \quad (3.2)$$

Upon complexifying $\mathcal{H}_{\mathbb{R}}$, real operators correspond exactly to complex-linear operators commuting with canonical conjugation $\mathcal{C}$. Consequently $\mathfrak{C}_U$ is canonically isomorphic to the Real commutant used in the published phase construction. The underlying operator and representation theory is standard [4, 5]; the emergence of complex structure under stronger relativistic hypotheses has also been studied by Moretti and Oppio [6]. Existence of J is a separate issue: for example, an odd-dimensional real space has no orthogonal complex structure.

3.2 Polar selection and its domain

Theorem 3.1 (Oriented phase selection). *Let $U(t) = e^{tK}$ be a strongly continuous orthogonal group with real skew-adjoint generator K . Set $\mathcal{H}_{\text{dyn}} = (\ker K)^\perp$ and $|K| = (-K^2)^{1/2}$. On $\mathcal{H}_{\text{dyn}}$, the polar factor J_K satisfies*

$$K = J_K|K| = |K|J_K, \quad J_K^* = -J_K, \quad J_K^2 = -\text{Id}, \quad \text{Dom } K = \text{Dom } |K|. \quad (3.3)$$

It is unique as the polar factor on that subspace. Any orthogonal symmetry commuting with the full flow commutes with J_K .

Proof. The skew-adjoint spectral theorem defines $|K|$ and the polar decomposition. Its initial space is the closure of $\text{Ran } |K|$, which is $(\ker K)^\perp$. For skew-adjoint normal K , the range support is the same, so the polar factor is unitary on the dynamic space. Spectral functional calculus gives $J_K^* = -J_K$ and $J_K^2 = -\text{Id}$ there. It also gives the commutation with $|K|$ and equality of domains. A symmetry commuting with e^{tK} commutes with the generator and its spectral functions, hence with the polar factor. $\square$

No gap around zero is needed. The expression $K|K|^{-1}$ is a convenient formal description on the spectral support, but $|K|^{-1}$ can be unbounded. The bounded polar phase is the correct definition. On $\ker K$ the polar factor is zero, so it is not a complex structure there without further data. Reversing the oriented flow sends J_K to $-J_K$. If one calls J_K multiplication by i , the group convention is $e^{+it|K|}$; the usual e^{-itH} Schrödinger convention uses the opposite phase or the opposite parameter orientation. No physical energy sign is inferred from a convention alone.

3.3 A phase is not an electromagnetic field

The selected J defines complex scalar multiplication. Its existence does not produce a local photon, a Maxwell kinetic term, an electric charge unit, or an electromagnetic coupling. A global transformation $e^{\theta J}$ may be only a scalar convention. Turning changes of local frame into a dynamical gauge field requires a bundle, transport, locality, a physical representation, and an action. These are the additional layers developed below.

Composition also needs care. For two selected real spaces, the raw real tensor product has twice the real dimension of the corresponding complex tensor product in finite dimensions. The complex composite is obtained by imposing

$$J_1 x \otimes y = x \otimes J_2 y \quad (3.4)$$

and completing the balanced tensor product with its complex Hilbert norm. This rule is additional compositional input [2, 7]. A one-particle phase theorem does not settle the full many-body composition law.

3.4 Phase selection can reduce quaternionic symmetry

Before an oriented phase is fixed, a real multiplicity algebra can contain quaternionic factors. Their unitary groups are symplectic. After the phase is fixed, the physically complex-linear symmetries must centralize it.

Proposition 3.2 (Quaternionic stabilizer). *Consider the standard quaternionic Hermitian space $\mathbb{H}^m$, with matrices acting on the left, and choose $J = iI_m$ for a fixed quaternionic imaginary unit. Then*

$$\{u \in \text{Sp}(m) : uJ = Ju\} \simeq U(m). \quad (3.5)$$

For $m = 1$, the stabilizer is $U(1)$, not $SU(2)$.

Proof. Write every quaternionic matrix uniquely as $u = P + Q\mathbf{j}$ with $P, Q \in M_m(\mathbb{C})$ and $\mathbf{j}\mathbf{i} = -\mathbf{i}\mathbf{j}$. Commutation with $\mathbf{i}I_m$ forces $Q = 0$. The quaternionic unitary condition then becomes $P^*P = I_m$, exactly the defining condition for $U(m)$. $\square$

Thus unit quaternions $\text{Sp}(1) \simeq SU(2)$ do not automatically survive as an internal weak group after a quaternionic phase direction is chosen. The algebra automorphism group $\text{Aut}(\mathbb{H}) \simeq SO(3)$ is another, different group. Spatial spin rotations, quaternionic frame rotations, and weak isospin must be assigned distinct actions.

One consistent possibility is to let complex phase act by right multiplication on $\mathbb{H}$, and let unit quaternions act on the left. The two commute. This exhibits a complex two-dimensional $SU(2)$ carrier, but the left/right separation is supplied, not selected by phase alone. For physical weak interactions that carrier must further act on left-handed matter while the right-handed matter is a singlet.

Proposition 3.3 (Multiplicity underdetermination). *For every $n \geq 1$, there is a real orthogonal flow with selected complex phase whose commuting orthogonal group is $U(n)$. Therefore oriented real time and phase selection alone do not fix the color multiplicity $n = 3$.*

Proof. Take $\mathcal{H}_{\mathbb{R}}$ to be the realification of $\mathbb{C}^n$, let J be multiplication by $\mathbf{i}$, and choose $K = \omega J$ with $\omega > 0$. Then $|K| = \omega I$ and $J_K = J$. The orthogonal maps commuting with all e^{tK} are precisely the complex-linear norm-preserving maps, namely $U(n)$. The construction works for every n . $\square$

Distinct nonzero frequencies on several blocks similarly give products of unitary groups. A determinant restriction may reduce $U(n)$ to $SU(n)$, but it does not select n . This counterexample is a direct test of any alleged derivation of the Standard Model from the phase theorem.

4 Transport on a completion network

4.1 Edges and changes of frame

Let $\mathcal{G}$ be a connected graph of completed records. Assign a finite-dimensional Hermitian space E_r to each record r , and a unitary map $L_{sr} : E_r \rightarrow E_s$ to each oriented edge $r \rightarrow s$, with $L_{rs} = L_{sr}^{-1}$. A section is a family $\psi_r \in E_r$. Local unitary frames u_r act by

$$\psi'_r = u_r \psi_r, \quad L'_{sr} = u_s L_{sr} u_r^{-1}. \quad (4.1)$$

Hence $\psi_s - L_{sr} \psi_r$ transforms by u_s , and its norm is invariant. The allowed frame group can be a subgroup of $U(E_r)$ selected by additional algebra, grading, and reality data.

Physical completion order may be acyclic. It then contains no closed future-directed paths. The inverse links in this graph are comparison maps between fibers, not reverse physical histories. Holonomy compares alternative transports between the same endpoints, including zigzags in a mathematical path groupoid. It must not be interpreted as a closed causal history.

4.2 Loops and curvature data

For a path $p : r_0 \rightarrow r_1 \rightarrow \cdots \rightarrow r_k$, let

$$L(p) = L_{r_k r_{k-1}} \cdots L_{r_1 r_0}. \quad (4.2)$$

If $r_k = r_0$, this is a holonomy endomorphism of E_{r_0} .

Proposition 4.1 (Gauge covariance of network holonomy). *For an open path, $L'(p) = u_{r_k} L(p) u_{r_0}^{-1}$. For a loop, its conjugacy class and traces $\text{tr } L(p)^k$ are frame invariant.*

Proof. Substitute (4.1) into the ordered product. The intermediate $u_r^{-1} u_r$ factors cancel. For a loop the two remaining factors conjugate the holonomy; traces are conjugation invariant. $\square$

In a unitary finite-dimensional representation, an identity holonomy has $\Re \text{tr } L = \dim E$. Conversely, equality of this real trace with $\dim E$ forces every unit-modulus eigenvalue to be one and thus $L = I$. A faithful representation is needed to infer the identity of the underlying group element.

Theorem 4.2 (Flat network criterion). *All loop holonomies in a connected graph are the identity if and only if local frames can make every edge transport the identity.*

Proof. If the links are identity, every holonomy is identity, and frame covariance proves necessity. Conversely choose a root r_* and a path p_r from r_* to each r . Set $g_r = L(p_r)$ and identify the fibers using those maps. Trivial loop holonomy makes transport path independent. Thus $L_{sr} = g_s g_r^{-1}$. Choosing $u_r = g_r^{-1}$ makes every transformed edge identity. $\square$

On a graph there is no intrinsic notion of an elementary plaquette unless a two-dimensional cell structure is supplied. Requiring identity only on boundaries of supplied faces gives a flat connection on that cell complex; noncontractible global loops can still carry holonomy. Accordingly local curvature zero and globally trivial transport are different statements.

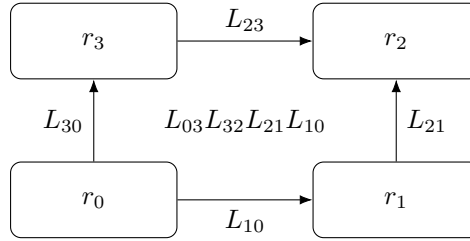


Figure 1: Two forward routes from r_0 to r_2 can be compared by an algebraic loop. Inverse comparison maps do not reverse physical completion order. A discrepancy is holonomy; identifying it with a force requires a physical representation and dynamics.

5 When does microscopic transport descend to records?

The completion map may identify many hidden histories. A transport defined before completion can depend on those distinctions. It therefore need not define a transport on observable records.

5.1 Descent through chosen fiber identifications

Let $q : X \rightarrow R$ be a surjective record map, with hidden fibers E_x and record fibers F_r . Suppose unitary identifications $I_x : E_x \rightarrow F_{q(x)}$ have been supplied. A microscopic arrow $a : x \rightarrow y$ carries $L_a : E_x \rightarrow E_y$ and projects to a specified record arrow $\bar{a} : q(x) \rightarrow q(y)$. Define

$$\hat{L}_a = I_y L_a I_x^{-1}. \quad (5.1)$$

Theorem 5.1 (Transport descent criterion). *A unique record transport $\bar{L}_{\bar{a}}$ satisfying $\bar{L}_{\bar{a}} = \hat{L}_a$ for every lift a exists exactly when*

$$\hat{L}_a = \hat{L}_b \quad \text{whenever } a, b \text{ project to the same record arrow.} \quad (5.2)$$

If microscopic composition projects to record composition, these maps preserve composition whenever the identifications are compatible at intermediate endpoints.

Proof. Necessity follows by comparing both lifts with the same proposed map. Sufficiency defines $\bar{L}_a$ using any lift, with independence guaranteed by (5.2). For composable arrows $a : x \rightarrow y$ and $b : y \rightarrow z$, one has $I_z L_b L_a I_x^{-1} = (I_z L_b I_y^{-1})(I_y L_a I_x^{-1})$. This is the required composition law. $\square$

This statement is elementary but physically decisive. Equality of all scalar record tests need not imply equality of transport on charged fibers. If the test family contains only gauge-invariant traces, transport is determined at best up to the appropriate equivalence. A record vector bundle then needs compatible local choices and cocycles, not a presumed global canonical identification.

5.2 Invisible vertical holonomy

For a microscopic loop that projects to an identity record arrow, exact descent requires identity action after identification. If physical states see only a representation $\rho : G \rightarrow U(F)$, the weaker condition is

$$\rho(L_{\text{vertical}}) = I. \quad (5.3)$$

Thus vertical holonomy can survive inside $\ker \rho$ without an observable effect. Conversely nontrivial holonomy acting on retained states obstructs exact autonomous transport descent. One must retain additional record labels, use a different quotient, or accept an effective description with memory or stochastic transport.

Example 5.2 (A projection that discards flux). On a square graph with one-dimensional fibers, set three links to 1 and the fourth to $e^{i\vartheta}$. Identify all four vertices with one record and project all edge transports to the identity arrow. Unless $\vartheta \in 2\pi\mathbb{Z}$, the square holonomy is nontrivial and cannot descend as identity transport on that one-dimensional record fiber. Discarding the loop label does not remove its phase action; it removes information needed for an autonomous description.

5.3 A continuum version

For a smooth surjective submersion $q : X \rightarrow M$, a connection pulled back from M has, in pullback-adapted gauges, a horizontal connection form $\mathcal{B} = q^*b$. In particular its curvature is $\mathcal{F} = q^*f$. A sufficient global descent datum is a bundle isomorphism $E \simeq q^*F$ together with a basic connection compatible with it. Vanishing curvature in vertical directions alone is not sufficient: flat vertical holonomy around noncontractible fibers may remain. This is the smooth counterpart of (5.3).

For DTT this separates two issues: a temporal curvature in generative variables may describe path-dependent memory, while a descended connection on spacetime describes gauge transport of retained states. They can have a common source in a specified model, but equality of the two curvatures is not automatic.

6 Smooth gauge geometry from local record transport

6.1 Continuum hypothesis

Hypothesis 6.1 (Smooth record and transport limit). A sequence of completed-record networks admits a smooth four-dimensional spacetime limit M , and its retained internal fibers assemble

into a smooth Hermitian bundle $E \rightarrow M$. In local charts the links converge to parallel transport of a differentiable connection. The allowed local frame group is a specified compact group G with representation ρ on E .

Neither the dimension of M nor G is proved by this hypothesis. Once it holds, the local form of the connection is forced by covariance.

Theorem 6.2 (Local gauge law and curvature). *If $\psi' = u\psi$ and $\nabla\psi$ is required to transform as $u\nabla\psi$, then*

$$\mathcal{B}' = u\mathcal{B}u^{-1} - (du)u^{-1}, \quad \mathcal{F}' = u\mathcal{F}u^{-1}. \quad (6.1)$$

In components,

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu + [\mathcal{B}_\mu, \mathcal{B}_\nu], \quad [\nabla_\mu, \nabla_\nu] = \mathcal{F}_{\mu\nu}. \quad (6.2)$$

Proof. Expanding $(d + \mathcal{B}')(u\psi) = u(d + \mathcal{B})\psi$ gives $(du) + \mathcal{B}'u = u\mathcal{B}$, hence the first law. The operator identity $(\nabla')^2 = u\nabla^2 u^{-1}$ gives the curvature law. Expanding ∇^2 on sections gives the component formula. $\square$

For a parallel section, transport over an infinitesimal displacement is $L(x + \delta x, x) = I - \mathcal{B}_\mu \delta x^\mu + O(|\delta x|^2)$. The sign of the infinitesimal loop is fixed by loop orientation; its leading deviation is $\pm \mathcal{F}_{\mu\nu} \delta x^\mu \delta x^\nu$.

6.2 Bianchi identity and dynamics

The graded Leibniz rule and $\mathcal{F} = d\mathcal{B} + \mathcal{B}^2$ give

$$d_\nabla \mathcal{F} = d\mathcal{F} + \mathcal{B} \wedge \mathcal{F} - \mathcal{F} \wedge \mathcal{B} = 0. \quad (6.3)$$

For a compact Lie algebra and an invariant positive inner product $\langle \cdot, \cdot \rangle$, a Riemannian Yang–Mills functional is

$$S_{\text{YM}}^E = \frac{1}{2g^2} \int_M \langle \mathcal{F} \wedge * \mathcal{F} \rangle. \quad (6.4)$$

Varying $\mathcal{B}$ gives $\delta \mathcal{F} = d_\nabla \delta \mathcal{B}$. After integration by parts on a closed manifold, stationarity gives $d_\nabla * \mathcal{F} = 0$ or a sourced equation with matter. Covariance determines the connection law and the commutator term, but it does not require this functional, determine g , or establish that the connection propagates. Those are dynamical choices.

6.3 Charge quantization and the correct electroweak interpretation

The continuous one-dimensional unitary characters of a circle are

$$\chi_n(e^{i\theta}) = e^{in\theta}, \quad n \in \mathbb{Z}. \quad (6.5)$$

Indeed a character lifted to $\mathbb{R}$ is $e^{iq\theta}$, and 2π periodicity forces $q \in \mathbb{Z}$. This quantizes representation weights relative to a chosen circle normalization. It does not fix the smallest physical charge, the coupling, or α ; higher integer weights can be absent or present depending on matter content. Temporal winding can motivate a compact phase group, but identifying winding with a charge representation is additional input.

Above electroweak symmetry breaking, the Abelian factor is hypercharge $U(1)_Y$, not electromagnetism. The photon is associated with the surviving generator $Q = T_3 + Y$. A proposed temporal $U(1)$ must specify which of these it represents. This distinction is necessary to avoid claiming a photon from scalar complex phase before the weak and Higgs sectors exist.

7 What is required to select the Standard Model algebra?

7.1 Finite real algebras admit many alternatives

A finite-dimensional real C^* -algebra has a direct-sum form

$$\mathcal{A}_F \simeq \bigoplus_a M_{r_a}(\mathbb{R}) \oplus \bigoplus_b M_{s_b}(\mathbb{C}) \oplus \bigoplus_c M_{t_c}(\mathbb{H}). \quad (7.1)$$

Its unitary factors are $O(r_a)$, $U(s_b)$, and $\mathrm{Sp}(t_c)$. These are standard algebraic possibilities [8, 9]; the matrix sizes and number of summands are not fixed by “real, complex, and quaternionic” alone. Equations (3.5) and the multiplicity counterexample show why a selected phase cannot repair this underdetermination by itself.

7.2 The candidate internal algebra

A familiar almost-commutative Standard Model construction uses the real algebra

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}). \quad (7.2)$$

Its algebra-unitary group is $U(1) \times SU(2) \times U(3)$. This is not yet exactly the physical gauge group. In a real spectral representation, the lifted gauge transformation has the form

$$U_u = \pi(u) \mathcal{J} \pi(u) \mathcal{J}^{-1}. \quad (7.3)$$

One must specify the bimodule, central lift, determinant or unimodularity restriction, and kernel. On the appropriate Standard Model module, these lead to the expected Lie algebra and a finite central quotient. On a different module they need not give the same hypercharges or quotient. Algebra, unitary group, lifted gauge action, and faithful physical group are four related but distinct objects.

Classification results in noncommutative geometry constrain the internal algebra through reality, grading, irreducibility, and first-order assumptions [12, 11]. These are valuable conditional results. Applying them to DTT requires deriving or justifying their hypotheses in a particular completion model; renaming a prescribed spectral triple “temporal” does not prove that DTT selects it.

7.3 Why the division-algebra ladder is insufficient

Unit quaternions provide $SU(2)$, but quaternion automorphisms provide $SO(3)$. Octonion automorphisms provide G_2 , with an $SU(3)$ stabilizer of a chosen unit imaginary direction [14]. Neither statement identifies a color carrier, a three-generation chiral module, or a dynamical color connection. Moreover $\mathbb{O}$ is nonassociative whereas (7.2) is associative. A nonassociative primitive model needs an explicit associative operator representation and a demonstration of how its constraints are transferred.

In particular S^7 is not a Lie group manifold analogous to $S^3 \simeq SU(2)$, and $S^3 \times S^3$ does not supply $SU(3)$ by a simple sphere-product rule. Compact spaces and their isometry groups can be used in separate model-building schemes, but those choices cannot be advertised as derived DTT consequences.

7.4 An explicit selection problem

Hypothesis 7.1 (Internal selection problem). There exists a specified class of primitive completion models and a stability or minimization rule, independent of the observed list of

particles, such that its selected finite representation is equivalent to a Standard Model finite geometry with algebra (7.2), its chiral bimodule, and an allowed finite Dirac operator.

This is an open target, not an assumption used to prove that DTT already unifies forces. A useful selection theorem would classify competing stable algebras and show why alternatives such as $\mathbb{C} \oplus \mathbb{H} \oplus M_n(\mathbb{C})$ with $n \neq 3$ fail. Minimality must be defined independently of the desired answer. Likewise three spatial coordinates do not force a three-dimensional color representation; the two groups act on different spaces.

8 Matter, chirality, hypercharge, and anomalies

8.1 The one-generation benchmark

Gauge groups acquire physical meaning through the representations of matter. We use only left-handed Weyl fields in the anomaly calculation. The conjugates of right-handed fields consequently have conjugate color representations and opposite hypercharge. With $Q = T_3 + Y$, one generation, optionally with a sterile neutrino, is

Field	$SU(3)_C$	$SU(2)_L$	Y	Multiplicity
q_L	3	2	$1/6$	6
u_R^c	$\overline{\mathbf{3}}$	1	$-2/3$	3
d_R^c	$\overline{\mathbf{3}}$	1	$1/3$	3
ℓ_L	1	2	$-1/2$	2
e_R^c	1	1	1	1
ν_R^c (optional)	1	1	0	1

Without the sterile state there are 15 Weyl components; with it there are 16. Three generations repeat the module three times. The Higgs is a complex scalar in $(\mathbf{1}, \mathbf{2})_{1/2}$ and does not contribute to fermion gauge anomalies.

A grading $\gamma_F^2 = 1$ distinguishes left and right internal sectors, but a grading alone does not determine which sector is an $SU(2)$ doublet. A DTT interpretation of chirality as oriented projection must construct a grading, its compatibility with the algebra, and the unequal left/right actions. Reversing an inner-time flow flips its selected phase; it does not by itself turn a weak doublet into a singlet.

8.2 Perturbative anomaly cancellation

Take the quadratic index of the fundamental of $SU(N)$ to be $T(\mathbf{N}) = 1/2$. For one generation the mixed anomalies are

$$\mathcal{A}_{SU(3)^2Y} = \frac{1}{2} \left(2 \cdot \frac{1}{6} - \frac{2}{3} + \frac{1}{3} \right) = 0, \quad (8.1)$$

$$\mathcal{A}_{SU(2)^2Y} = \frac{1}{2} \left(3 \cdot \frac{1}{6} - \frac{1}{2} \right) = 0. \quad (8.2)$$

The cubic hypercharge and mixed gravitational-hypercharge anomalies are

$$\mathcal{A}_{Y^3} = 6 \left(\frac{1}{6} \right)^3 + 3 \left(-\frac{2}{3} \right)^3 + 3 \left(\frac{1}{3} \right)^3 + 2 \left(-\frac{1}{2} \right)^3 + 1 \quad (8.3)$$

$$= \frac{1}{36} - \frac{8}{9} + \frac{1}{9} - \frac{1}{4} + 1 = 0, \quad (8.4)$$

$$\mathcal{A}_{\text{grav}^2Y} = 6 \left(\frac{1}{6} \right) + 3 \left(-\frac{2}{3} \right) + 3 \left(\frac{1}{3} \right) + 2 \left(-\frac{1}{2} \right) + 1 = 0. \quad (8.5)$$

The $SU(3)^3$ anomaly cancels because the two fundamentals in q_L are balanced by the two anti-fundamentals u_R^c, d_R^c . There is no local cubic $SU(2)$ anomaly for these pseudoreal doublets. Mixed anomalies with one non-Abelian generator and two Abelian generators vanish by tracelessness of that generator.

8.3 Global weak anomaly

For fundamental $SU(2)$ doublets, the Witten global anomaly requires an even number of left-handed doublets [21]. Here the three color copies of q_L and the single lepton doublet give $3 + 1 = 4$. Repetition through three generations gives 12, still even. This checks the familiar global weak anomaly on ordinary spin backgrounds. It does not classify every possible global anomaly for all quotient groups, non-spin backgrounds, or added matter; such extensions need their own analysis.

8.4 A conditional derivation of hypercharge ratios

Within a fixed matter ansatz, anomaly cancellation is more than a check. It can constrain relative charges.

Proposition 8.1 (Hypercharge ratios with minimal matter). *Assume the non-Abelian representations of the five nonsterile fields in the table, one Higgs doublet, all three charged-fermion Yukawa couplings, no right-handed neutrino, and cancellation of mixed gravitational hypercharge and $SU(2)^2Y$ anomalies. If $Y(q_L) = y$, then*

$$Y(\ell_L) = -3y, \quad Y(H) = 3y, \quad Y(u_R^c) = -4y, \quad Y(d_R^c) = 2y, \quad Y(e_R^c) = 6y. \quad (8.6)$$

These values also cancel the remaining perturbative anomalies. The overall normalization y remains free.

Proof. Write $Y(H) = h$. Gauge invariance of the two-component Yukawa terms gives $Y(u_R^c) = -y - h$, $Y(d_R^c) = -y + h$, and $Y(e_R^c) = -Y(\ell_L) + h$. The weak mixed anomaly gives $Y(\ell_L) = -3y$. The mixed gravitational anomaly is then

$$6y + 3(-y - h) + 3(-y + h) + 2(-3y) + (3y + h) = h - 3y. \quad (8.7)$$

Its vanishing implies $h = 3y$. Substitution yields (8.6). The color mixed anomaly vanishes by $2y + (-4y) + 2y = 0$, and the cubic anomaly is proportional to $6 - 192 + 24 - 54 + 216 = 0$. $\square$

This proposition derives ratios given a representation and Yukawa ansatz; it does not derive that ansatz from ticks. If a sterile neutrino and its Dirac Yukawa term are allowed, $Y(\nu_R^c) = 3y - h$. The mixed gravitational anomaly then cancels for arbitrary h , and the cubic anomaly also cancels. An extra anomaly-free Abelian direction associated with $B - L$ is possible. Requiring the sterile field to be neutral or allowing a gauge-invariant Majorana mass sets $h = 3y$ again. Thus even anomaly cancellation does not uniquely select the Abelian structure without the stated matter assumptions.

8.5 The finite central quotient

Normalize the circle so that its integer weight is $q = 6Y$. Let $\omega = e^{2\pi i/3}$. On the benchmark representations the element

$$z = (\omega I_3, -I_2, e^{i\pi/3}) \quad (8.8)$$

acts trivially. For example, on q_L its phase is $\omega(-1)e^{i\pi/3} = 1$, on u_R^c it is $\omega^{-1}e^{-4i\pi/3} = 1$, and on ℓ_L it is $(-1)e^{-3i\pi/3} = 1$. It also acts trivially on the Higgs of weight 3.

Proposition 8.2 (Faithful benchmark group). *The kernel of the product-group action on these representations is the cyclic group generated by z , of order six. Therefore its faithful image is*

$$G_{\text{SM}}^{\text{faithful}} = \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6}. \quad (8.9)$$

Proof. The quark and lepton representations force any kernel element to be scalar in the two non-Abelian factors. Write it as $(\omega^a, (-1)^b, e^{i\theta})$. The charged singlet implies $6\theta \in 2\pi\mathbb{Z}$, so $\theta = k\pi/3$. The lepton doublet gives $b = k \pmod{2}$, while d_R^c gives $a = k \pmod{3}$. The remaining representations then impose no further restriction, so the kernel is exactly the six powers of z . $\square$

Other global forms can be considered when only the Lie algebra and local matter couplings are specified; the allowed bundles and line operators then differ. The quotient above is a result for this faithful matter action, not a universal deduction from the symbols $SU(3), SU(2), U(1)$.

9 Finite spectral transitions and the Higgs sector

9.1 The finite geometry

A finite real even spectral geometry consists of

$$(\mathcal{A}_F, \mathcal{H}_F, D_F, \mathcal{J}_F, \gamma_F). \quad (9.1)$$

Here $\mathcal{A}_F$ is a represented real $*$ -algebra, $\mathcal{H}_F$ is complex after phase selection, $D_F = D_F^*$, and γ_F is a self-adjoint grading commuting with the algebra and anticommuting with D_F . The antiunitary $\mathcal{J}_F$ implements the opposite action. The order-zero and first-order conditions are

$$[\pi(a), \pi(b)^0] = 0, \quad [[D_F, \pi(a)], \pi(b)^0] = 0, \quad \pi(b)^0 = \mathcal{J}_F \pi(b)^* \mathcal{J}_F^{-1}. \quad (9.2)$$

An allowed sign convention for the usual finite KO-dimension-six module is

$$\mathcal{J}_F^2 = 1, \quad \mathcal{J}_F D_F = D_F \mathcal{J}_F, \quad \mathcal{J}_F \gamma_F = -\gamma_F \mathcal{J}_F. \quad (9.3)$$

These signs and conditions are mathematical input to that spectral construction [11]. They are not consequences of the existence of J_K .

9.2 Off-diagonal transitions

On a simple left/right sector the self-adjoint finite operator has the form

$$D_F = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}. \quad (9.4)$$

A finite one-form $a[D_F, b]$ changes the transition block. If M links inequivalent representations, its fluctuation transforms as a map between them. For example, if $M : E_R \rightarrow E_L$, covariance is

$$M(x) \mapsto u_L(x) M(x) u_R(x)^{-1}. \quad (9.5)$$

Thus an off-diagonal fluctuation can be a scalar in a nontrivial bifundamental representation. For the usual Standard Model finite module, the charged transition sector supplies a complex Higgs doublet. For another module it may supply several scalars or no Higgs doublet at all.

Example 9.1 (A two-sector toy geometry). Let $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{C}$ act diagonally on $\mathbb{C}^2$, with the grading $\text{diag}(1, -1)$ and D_F as in (9.4) with scalar $M = m$. Then

$$[D_F, \text{diag}(b_1, b_2)] = \begin{pmatrix} 0 & m(b_2 - b_1) \\ \bar{m}(b_1 - b_2) & 0 \end{pmatrix}. \quad (9.6)$$

The transition fluctuation is a complex off-diagonal scalar. This illustrates the mechanism without pretending to derive a weak doublet. A complete real spectral triple would additionally require a specified opposite representation and verification of (9.2); the example is only the transition calculation.

9.3 Yukawa couplings and masses

With physical right-handed fields, the Standard Model Yukawa sector is

$$\mathcal{L}_Y = -\bar{q}_L \tilde{H} Y_u u_R - \bar{q}_L H Y_d d_R - \bar{\ell}_L H Y_e e_R + \text{h.c.}, \quad (9.7)$$

$$\tilde{H} = i\sigma_2 H^*, \quad (9.8)$$

optionally adding $-\bar{\ell}_L \tilde{H} Y_\nu \nu_R$ and allowed sterile Majorana terms. At

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad (9.9)$$

the charged masses are singular values of $vY_f/\sqrt{2}$. Diagonalization of these matrices gives flavor mixing. An assumed D_F can encode these values, but encoding measured Yukawa matrices is not predicting them.

The eigenvalues of (9.4) are $\pm s_j(M)$, the signed singular values of M . They need not be the frequencies of the microscopic generator K . If a model explicitly identifies a rest-frame frequency ω_j with a mass mode, then $E_j = \hbar\omega_j$ and $E_j = m_j c^2$ give

$$m_j = \frac{\hbar\omega_j}{c^2}. \quad (9.10)$$

This conversion relation does not determine ω_j , $\hbar$, or m_j . If instead D_F has inverse-length eigenvalues λ_j , the corresponding conversion is $m_j = \hbar\lambda_j/c$. Confusing these two units can conceal a factor of c and an undetermined scale.

10 Spectral geometry and a common action

10.1 Product data

For the heat-kernel analysis assume a compact oriented Riemannian spin four-manifold M_E without boundary, with spinor bundle S . The standard product data are

$$\mathcal{A} = C^\infty(M_E) \otimes_{\mathbb{R}} \mathcal{A}_F, \quad (10.1)$$

$$\mathcal{H} = L^2(M_E, S) \otimes_{\mathbb{C}} \mathcal{H}_F, \quad (10.2)$$

$$D = D_{M_E} \otimes 1 + \gamma_5 \otimes D_F, \quad (10.3)$$

$$\gamma = \gamma_5 \otimes \gamma_F. \quad (10.4)$$

The product real structure is chosen with the appropriate KO signs. Crucially, D is a graded sum: the γ_5 in (10.4) is needed for the correct cross-term cancellation. The informal sum $D_{\text{spacetime}} + D_{\text{inner}}$ is insufficient without a grading and tensor representation.

10.2 Inner fluctuations

For a self-adjoint represented one-form

$$\mathcal{A}_1 = \sum_j \pi(a_j)[D, \pi(b_j)], \quad (10.5)$$

the real fluctuation, under the first-order condition, is

$$D_{\mathcal{A}_1} = D + \mathcal{A}_1 + \varepsilon' \mathcal{J} \mathcal{A}_1 \mathcal{J}^{-1}, \quad (10.6)$$

with the sign ε' fixed by the real-structure convention. Its smooth components are gauge one-forms; its finite components are scalar transition fields. Removing the first-order condition can require additional quadratic fluctuation terms and changes the model, so (10.6) is not a universal formula for every noncommutative geometry.

For a fixed D_{M_E} , internal inner fluctuations do not freely generate an arbitrary metric. The metric already occurs in the principal symbol of D_{M_E} . Gravitation enters through the dependence of the action on that geometric Dirac operator and variation of the metric data. A fundamental DTT account must obtain these metric data from records and make them dynamical, rather than claim that an internal gauge fluctuation alone creates spacetime.

10.3 The spectral action and physical fermions

Use a nonnegative sufficiently rapidly decreasing profile f and a cutoff Λ of inverse-length dimension. The bosonic action is

$$S_B = \text{Tr } f(D_{\mathcal{A}_1}^2/\Lambda^2). \quad (10.7)$$

The trace is over the full spectral Hilbert space. The fermionic term requires the physical chiral restriction and the real structure; in the standard real formulation its bilinear has the schematic form

$$S_F = \frac{1}{2} \langle \mathcal{J} \Psi, D_{\mathcal{A}_1} \Psi \rangle, \quad \Psi \in \mathcal{H}_{\text{physical}}^+, \quad (10.8)$$

with Grassmann variables and a convention-dependent Euclidean continuation. Integrating all copies in a naive $\langle \Psi, D\Psi \rangle$ on the doubled Hilbert space would not automatically recover the physical fermion content. The doubling issue and the relevant reality prescription are part of the known spectral Standard Model construction [11].

10.4 Heat-kernel coefficients

Write the Laplace-type square locally as

$$P = D_{\mathcal{A}_1}^2 = - \left[g^{\mu\nu} (\nabla_\mu \nabla_\nu - \Gamma_{\mu\nu}^\rho \nabla_\rho) + \mathcal{E} \right]. \quad (10.9)$$

Let $\Omega_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$. On a closed four-manifold,

$$\text{Tr } e^{-tP} \sim t^{-2} a_0(P) + t^{-1} a_2(P) + a_4(P) + \cdots, \quad (10.10)$$

where the first three integrated coefficients are [13]

$$a_0(P) = \frac{1}{(4\pi)^2} \int \sqrt{g} \text{tr } I \, d^4x, \quad (10.11)$$

$$a_2(P) = \frac{1}{(4\pi)^2} \int \sqrt{g} \text{tr} (\mathcal{E} + R/6) \, d^4x, \quad (10.12)$$

$$a_4(P) = \frac{1}{360(4\pi)^2} \int \sqrt{g} \text{tr} \left[60R\mathcal{E} + 180\mathcal{E}^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu} + (5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})I \right] d^4x. \quad (10.13)$$

Integrated total derivatives have been omitted because the manifold is closed. Boundaries require boundary conditions and additional coefficients, not deletion by convention.

Define the moments for the squared-operator profile by

$$F_4 = \int_0^\infty f(u)u \, du, \quad F_2 = \int_0^\infty f(u) \, du, \quad F_0 = f(0). \quad (10.14)$$

Under the standard smooth elliptic asymptotic hypotheses,

$$S_B \sim F_4 \Lambda^4 a_0(P) + F_2 \Lambda^2 a_2(P) + F_0 a_4(P) + \cdots. \quad (10.15)$$

These moments differ in notation from those of a profile written as $\text{Tr } f(|D|/\Lambda)$. The definition above fixes the convention and prevents untracked factors of two.

10.5 An explicit gravitational coefficient

For a massless untwisted Dirac operator tensored with N_F internal states, the Lichnerowicz identity gives $\mathcal{E} = -R/4$. Since the four-dimensional spinor rank is four,

$$a_2(D^2) = -\frac{N_F}{48\pi^2} \int \sqrt{g} R \, d^4x. \quad (10.16)$$

Thus in this pure-gravity benchmark

$$S_B \supset -\frac{N_F F_2 \Lambda^2}{48\pi^2} \int \sqrt{g} R \, d^4x. \quad (10.17)$$

Matching the Euclidean Einstein term $-M_{\text{red}}^2 \int \sqrt{g} R/2$ gives

$$M_{\text{red}}^2 = \frac{N_F F_2 \Lambda^2}{24\pi^2}, \quad G = \frac{1}{8\pi M_{\text{red}}^2} = \frac{3\pi}{N_F F_2 \Lambda^2}. \quad (10.18)$$

This is a conditional coefficient match, not a prediction of Newton's constant. Finite masses, Higgs curvature couplings, and threshold corrections modify the gravitational matching; they must be included in a realistic model. The a_0 term also produces a contribution of order Λ^4 to vacuum energy, which is not solved by the existence of the action.

10.6 An explicit gauge coefficient

On a flat background with a massless twisted Dirac operator,

$$\mathcal{E} = \frac{1}{2} \gamma^{\mu\nu} \mathcal{F}_{\mu\nu}, \quad \Omega_{\mu\nu} = \mathcal{F}_{\mu\nu}. \quad (10.19)$$

The spin trace identities give

$$\text{tr}_S \mathcal{E}^2 = -2 \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}, \quad \text{tr}_S \Omega_{\mu\nu} \Omega^{\mu\nu} = 4 \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}. \quad (10.20)$$

Consequently (10.13) contains

$$a_4(P) \supset -\frac{1}{24\pi^2} \int \text{tr}_F (\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}) \, d^4x. \quad (10.21)$$

The minus sign makes this positive for anti-Hermitian curvature. The same result follows in curved space for the pure gauge density; mixed terms are handled by the full coefficient. Representation traces now determine relative gauge normalization, as calculated in Section 13.

10.7 What the finite transitions add

A scalar transition Φ in $D_{\mathcal{A}_1}$ contributes powers of Φ , covariant derivatives of Φ , and curvature couplings to $\mathcal{E}$ and (10.13). The resulting densities include

$$|D_\mu H|^2, \quad -\mu_H^2 H^\dagger H, \quad \lambda(H^\dagger H)^2, \quad RH^\dagger H, \quad (10.22)$$

for the standard finite module after canonical normalization. The scalar potential depends on the finite transition matrices and spectral moments. A stable Higgs vacuum requires the appropriate signs and thresholds; no vacuum is selected merely by writing a block matrix.

The effective action also contains curvature-squared densities and higher-order operators. Keeping only Einstein gravity plus renormalizable particle physics is an approximation with a scale hierarchy. Treating a finite higher-derivative truncation as an exact ultraviolet theory can introduce additional poles, so its healthy low-energy interpretation does not establish a unitary fundamental completion.

10.8 The physical effective-action target

After canonical normalization and a justified physical continuation, the target action in signature $(-, +, +, +)$ can be displayed explicitly as

$$S_{\text{eff}} = \int_M \sqrt{-g} d^4x \left[\frac{M_{\text{red}}^2}{2} (R - 2\Lambda_{\text{cos}}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W_{\mu\nu}^a W^{a\mu\nu} - \frac{1}{4} G_{\mu\nu}^A G^{A\mu\nu} - (D_\mu H)^\dagger D^\mu H - V(H) - \xi R H^\dagger H + \mathcal{L}_{\text{fermion}} + \mathcal{L}_Y + \mathcal{L}_{\text{higher}} \right]. \quad (10.23)$$

Here $V(H) = -\mu_H^2 H^\dagger H + \lambda(H^\dagger H)^2$ in the simplest symmetry-breaking regime, $\mathcal{L}_{\text{fermion}}$ is the sum of covariant chiral fermion kinetic terms, and $\mathcal{L}_Y$ is the transition sector in Section 9. The field strengths use the canonically normalized couplings in their nonlinear and matter terms. For a field in representation $(R_3, R_2)_Y$,

$$D_\mu = \nabla_\mu^{\text{spin}} - ig_Y Y B_\mu - ig_2 T^a W_\mu^a - ig_3 t^A G_\mu^A, \quad (10.24)$$

with the absent generators acting by zero. Higher densities include curvature-squared terms, possible additional scalar terms, and operators suppressed in the stipulated low-energy domain. The nonminimal coefficient ξ is part of the geometric matching and affects the effective gravitational coefficient in a scalar background. Thus (10.23) includes assumptions beyond a minimal Einstein-plus-Standard-Model Lagrangian; setting every extra term to zero would require justification.

The desired microscopic statement is that a particular completion action converges to (10.23) with calculable coefficients and controlled corrections. Writing this action as a target, or reconstructing it from prescribed almost-commutative data, does not yet establish that microscopic statement.

11 Record geometry and the gravitational sector

11.1 A conditional cone from completion delays

Suppose a completed transition is built from displacements δx_j with nonnegative length budgets ℓ_j and delay budgets θ_j . Assume

$$\|\delta x_j\| \leq \ell_j, \quad \ell_j \leq c_* \theta_j, \quad \Delta x = \sum_j \delta x_j, \quad \Theta = \sum_j \theta_j. \quad (11.1)$$

Then the triangle inequality gives

$$\|\Delta x\| \leq \sum_j \ell_j \leq c_* \Theta. \quad (11.2)$$

This is a valid cone lemma given a Euclidean readout and a delay law. It does not derive c_* , select three dimensions, or show that every vector in the cone is attainable. The same proof works for $\mathbb{R}^d$ for arbitrary d . Finite propagation in discrete quantum systems also has established conditional forms, including Lieb–Robinson bounds [17, 18]; such bounds are usually estimates with tails, not by themselves exact continuum Lorentz invariance.

If the full readout cone is identified with $t \geq 0$, $c_*^2 t^2 - \|x\|^2 \geq 0$, and its smooth structure is given, its null boundary fixes a Lorentzian conformal class. A volume or clock calibration is additionally needed to fix the conformal factor. The role of causal relations in determining conformal geometry is a classical result with nontrivial causal hypotheses [19]. Recovering causal order is therefore a kinematic step, not the Einstein field equation.

11.2 Metric and tetrad data

Assume a smooth time-oriented Lorentzian spin manifold M arises in the record limit, with a coframe $e^a = e^a_\mu dx^\mu$ and

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu. \quad (11.3)$$

The spin connection ω^a_b has curvature and torsion

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b, \quad T^a = de^a + \omega^a_b \wedge e^b. \quad (11.4)$$

The torsion-free condition selects the Levi-Civita spin connection for a specified coframe. In first-order gravity with fermions, independent connection variation can instead produce torsion and effective contact terms. A spectral formulation using a Levi-Civita operator and a Palatini formulation with an independent connection are not identical inputs.

11.3 Einstein dynamics as an effective restriction

At the lowest derivative orders, a local generally covariant metric action with no additional light gravitational fields has terms

$$S_g = \frac{M_{\text{red}}^2}{2} \int \sqrt{-g} (R - 2\Lambda_{\text{cos}}) d^4x + S_{\text{higher}}. \quad (11.5)$$

The coefficient M_{red}^2 and cosmological constant remain dynamical matching data. Restricting to second-order metric field equations in four dimensions strongly constrains the admissible purely metric tensors [20]. This observation does not select a microscopic completion law or determine the gravitational coupling.

With the usual metric boundary term or suitable compactly supported variations,

$$\delta \int \sqrt{-g} R d^4x = \int \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} d^4x + \text{boundary contribution}. \quad (11.6)$$

Define $T_{\mu\nu} = -2(\sqrt{-g})^{-1} \delta S_m / \delta g^{\mu\nu}$. Stationarity gives

$$G_{\mu\nu} + \Lambda_{\text{cos}} g_{\mu\nu} = M_{\text{red}}^{-2} T_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (11.7)$$

up to the neglected higher-order terms. Diffeomorphism invariance and matter equations imply the corresponding stress-energy conservation. This calculation recovers the Einstein equation once (11.5) is supplied or matched; it is not a proof that a completion-order network has that action in its scaling limit.

11.4 Why an ordinary compact timelike dimension is problematic

Consider, only for comparison, a five-dimensional metric with two time directions,

$$ds^2 = -dt^2 + d\mathbf{x}^2 - dy^2, \quad y \sim y + 2\pi R. \quad (11.8)$$

For a scalar with five-dimensional mass m_5 , the equation $(\square_5 - m_5^2)\phi = 0$ and the mode $e^{iny/R}$ give

$$(\square_4 - m_{n,\text{eff}}^2)\phi_n = 0, \quad m_{n,\text{eff}}^2 = m_5^2 - \frac{n^2}{R^2}. \quad (11.9)$$

Arbitrarily high harmonics have negative effective mass squared. The compact y circles are also timelike closed curves of the ordinary five-dimensional metric. A timelike Kaluza–Klein reduction can produce wrong-sign vector kinetic terms. Merely making R small does not repair these problems. Constraints, a projection that removes the dangerous modes, or a different non-spacetime meaning of the parameter would have to be specified and proved consistent.

For an ordinary product reduction with physical internal volume V and convention $S_D = (2\kappa_D^2)^{-1} \int \sqrt{|g_D|} R_D$, constant-volume matching is

$$\frac{1}{\kappa_4^2} = \frac{V}{\kappa_D^2}, \quad \kappa_4^2 = 8\pi G. \quad (11.10)$$

This contains no extra factor of two. It relates two undetermined couplings and a supplied volume. It cannot determine G from first principles. The present framework consequently treats inner-generative time as the parameter of a completion or operator process, as in the published phase construction, and does not assume the five-dimensional metric used in (11.9).

12 Generalized transport: common curvature and its limits

On a spinor bundle tensored with an internal bundle, the combined covariant derivative is

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} \otimes 1 + 1 \otimes \mathcal{B}_\mu. \quad (12.1)$$

The actions on the two tensor factors commute, so

$$[\nabla_\mu, \nabla_\nu] = \frac{1}{4}R_{\mu\nu}^{ab}\gamma_{ab} \otimes 1 + 1 \otimes \mathcal{F}_{\mu\nu}. \quad (12.2)$$

When the internal representation separates the three gauge factors, $\mathcal{F}$ decomposes into hypercharge, weak, and color curvatures. Thus a single transport operator acts on matter with gravitational and gauge components. This is a precise common-description statement.

It is not sufficient to square (12.2) and claim the Einstein action. A curvature-square action for the spin connection generally contains $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and is not the Einstein–Hilbert action, which is linear in scalar curvature and requires the coframe or metric. Moreover the Lorentz group is noncompact, so a compact-gauge positivity argument cannot be copied into Lorentzian spin-connection dynamics without modification. The heat-kernel term (10.17) is the mechanism supplying the linear curvature density in the conditional spectral route.

12.1 Holonomy and observable effects

For a small comparison loop, the combined holonomy measures the curvature in (12.2). But nonzero curvature does not imply an observable force on every state. A neutral state may lie in a trivial representation; a vertical holonomy may be in the kernel of all retained representations;

a pure phase may be unobservable in a particular experiment. Effects require a coupling to matter and a specified observable protocol. For example, a colorless lepton does not acquire color acceleration simply because a color connection exists.

Accordingly the DTT phrase “interaction as curvature of reconstruction” is a proposed physical interpretation of a transport law. Its mathematical content is the connection and curvature formula once a retained bundle exists. Its force-law content comes from a matter action, and its ontology-to-physics content comes from a demonstrated completion model.

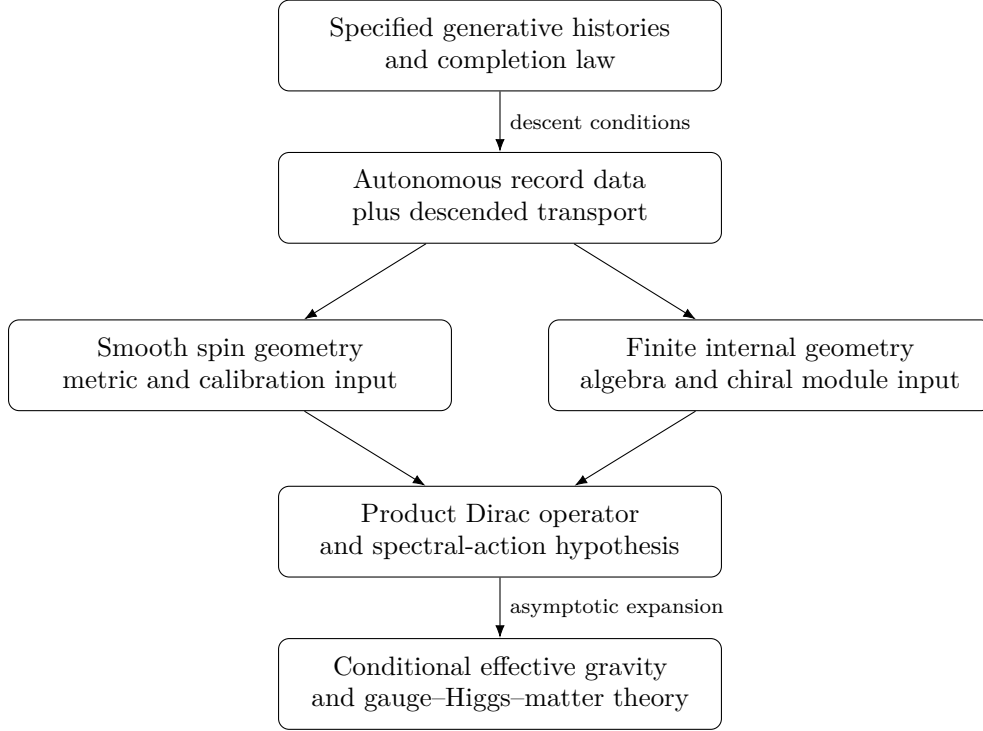


Figure 2: The dependency architecture. The branch labels explicitly include the supplied geometric data. The arrows do not assert that a primitive tick law has already been proved to select either branch.

13 Coupling constants, normalization, and scale matching

13.1 Representation traces

Let $T_a^{(i)}$ be Hermitian generators in the retained internal module, and define

$$\text{tr}_F(T_a^{(i)} T_b^{(i)}) = K_i \delta_{ab}. \quad (13.1)$$

For the Abelian factor the single generator is Y , so $K_Y = \text{tr}_F Y^2$. Using (10.21), $\mathcal{F} = -ig_i F_i^a T_a^{(i)}$ gives the Euclidean density

$$\frac{F_0 K_i g_i^2}{24\pi^2} F_{i\mu\nu}^a F_i^{a\mu\nu}. \quad (13.2)$$

Matching the canonical $\frac{1}{4} F_i^a F_i^a$ yields

$$g_i^{-2} = \frac{F_0 K_i}{6\pi^2}. \quad (13.3)$$

Equivalently, if the coupling is not included in the connection, its inverse square multiplies the unnormalized curvature kinetic term. These two conventions must not be mixed.

For one generation of physical Weyl fields, the trace weights are

$$K_3 = 2 \left(\frac{1}{2} \right) + \frac{1}{2} + \frac{1}{2} = 2, \quad (13.4)$$

$$K_2 = 3 \left(\frac{1}{2} \right) + \frac{1}{2} = 2, \quad (13.5)$$

$$K_Y = 6 \left(\frac{1}{6} \right)^2 + 3 \left(\frac{2}{3} \right)^2 + 3 \left(\frac{1}{3} \right)^2 + 2 \left(\frac{1}{2} \right)^2 + 1 = \frac{10}{3}. \quad (13.6)$$

Including the charge-conjugate copies doubles all three weights, and three generations multiply all by three. The ratio is unchanged:

$$K_Y : K_2 : K_3 = \frac{5}{3} : 1 : 1. \quad (13.7)$$

A sterile neutrino contributes to N_F in the gravitational trace but not to these gauge traces. This is a concrete reason why the gravitational and gauge normalizations are not simply reciprocals of one undifferentiated “inner-time volume.”

13.2 Conditional unification relation

With a single spectral moment and the benchmark representation,

$$g_3^2 = g_2^2 = \frac{5}{3} g_Y^2 \quad \text{at the matching scale } \mu = \Lambda. \quad (13.8)$$

Define $g_1 = \sqrt{5/3} g_Y$ to obtain equality in conventional GUT normalization. Equality of g_Y, g_2, g_3 without this normalization is not the corresponding spectral relation. Equation (13.8) is a representation-trace prediction conditional on a common action profile and supplied matter. It is not a numerical prediction for any one coupling until F_0 is fixed, and it can be altered by additional representations or nonuniversal weights.

13.3 Electroweak mixing

The covariant-derivative norm for a Higgs of hypercharge 1/2 contains at its vacuum

$$|D_\mu H|^2 \supset \frac{v^2}{8} (g_2 W_\mu^3 - g_Y B_\mu)^2. \quad (13.9)$$

The orthogonal massless and massive combinations are

$$A_\mu = \sin \theta_W W_\mu^3 + \cos \theta_W B_\mu, \quad (13.10)$$

$$Z_\mu = \cos \theta_W W_\mu^3 - \sin \theta_W B_\mu, \quad (13.11)$$

$$\tan \theta_W = g_Y / g_2, \quad e = g_2 \sin \theta_W = g_Y \cos \theta_W. \quad (13.12)$$

Therefore

$$\frac{1}{e^2} = \frac{1}{g_2^2} + \frac{1}{g_Y^2}, \quad \alpha_{\text{em}} = \frac{e^2}{4\pi}. \quad (13.13)$$

The benchmark high-scale relation implies $\sin^2 \theta_W(\Lambda) = 3/8$. It does not imply this value at laboratory scales. It also shows why a putative direct temporal determination of α must address both the weak and hypercharge sectors and their running.

13.4 Renormalization-group evolution

For the minimal three-generation Standard Model with one Higgs doublet, the one-loop coefficients in GUT normalization are

$$(b_1, b_2, b_3) = \left(\frac{41}{10}, -\frac{19}{6}, -7 \right), \quad \mu \frac{dg_i}{d\mu} = \frac{b_i}{16\pi^2} g_i^3. \quad (13.14)$$

Thus

$$g_i^{-2}(\mu) = g_i^{-2}(\Lambda) - \frac{b_i}{8\pi^2} \log \frac{\mu}{\Lambda} + \Delta_i^{\text{threshold}} + \Delta_i^{\text{higher}}. \quad (13.15)$$

For reference, the coefficients follow from

$$b = -\frac{11}{3}C_2(G) + \frac{2}{3} \sum_{\text{Weyl}} T(R) + \frac{1}{3} \sum_{\text{complex scalar}} T(R). \quad (13.16)$$

The color sum is $3 \cdot 2 = 6$, giving $-11 + 4 = -7$; the weak sum is also 6 and the Higgs adds $1/6$, giving $-22/3 + 4 + 1/6 = -19/6$. For normalized hypercharge, the Weyl sum is $(3/5) \cdot 3 \cdot (10/3) = 6$, while the Higgs scalar sum is $(3/5) \cdot 2 \cdot (1/2)^2 = 3/10$, giving $4 + 1/10 = 41/10$. These are perturbative coefficients for that matter ansatz, not an assertion that the minimal model realizes exact high-scale unification.

Any phenomenological DTT model must select the matching scale, additional thresholds, and renormalization prescription. A low-energy electromagnetic number cannot be equated to a high-scale unified boundary condition without this calculation.

13.5 The integer 137 and an explicit underdetermination test

The arithmetic identity

$$\sum_{d=0}^3 (2^d - 1)10^{3-d} = 0 + 100 + 30 + 7 = 137 \quad (13.17)$$

is exact. Identifying it with an inverse coupling requires a rule selecting the decimal weights, the four levels, the representation normalization, and the relevant physical scale. Neither the identity nor the winding group supplies the spectral moment F_0 in (13.3). Rescaling $f \mapsto af$ with $a > 0$ preserves the internal algebra, phase, and transport laws but rescales F_0 and hence changes all canonically matched gauge couplings.

Similarly, no correction from 137 to the measured low-energy inverse electromagnetic coupling follows just from inserting a small fractional adjustment. A predictive calculation must supply its dynamics and its renormalization scheme. This paper does not cite an unpublished proposal for that arithmetic identity or assert it as a derived physical constant.

13.6 Gravitational dimensions and remaining freedom

Equations (10.18) and (13.3) involve two different spectral moments, F_2 and F_0 , as well as a scale Λ . In the massless benchmark,

$$\alpha_i \frac{M_{\text{red}}^2}{\Lambda^2} = \frac{N_F F_2}{16\pi F_0 K_i}. \quad (13.18)$$

The left-hand side is dimensionless. The product $\alpha_i M_{\text{red}}^2$ alone has mass dimension two and cannot be a dimensionless universal constant without a specified scale. Furthermore the ratio F_2/F_0 is not fixed merely by using a common profile. Changing the shape of f at fixed $f(0)$ changes F_2 independently of the gauge normalization.

With Λ explicitly an inverse length, restoring SI units in the pure-gravity benchmark gives

$$\frac{\hbar G}{c^3} = \frac{3\pi}{N_F F_2 \Lambda^2}, \quad G = \frac{3\pi c^3}{N_F F_2 \hbar \Lambda^2}. \quad (13.19)$$

This formula displays rather than removes the undetermined scale and action normalization. A numerical prediction in SI units needs an independently established conversion between the primitive process and physical clocks, lengths, and action. Calling a tick “Planck-scale” using a scale already defined from G and $\hbar$ would be circular if the goal were to derive those constants.

14 A conditional assembly theorem

The preceding modules can be assembled without pretending that assembly proves their inputs.

Theorem 14.1 (Conditional temporal–spectral reconstruction). *Assume all of the following data and compatibility conditions.*

- (A1) *A history action, completion map, and test family define a stable behavioral quotient, as in Section 2. A positive Hilbertization and an oriented orthogonal flow are supplied, with a selected phase on the retained dynamic sector and explicit treatment of any zero sector.*
- (A2) *Retained internal transports satisfy the descent criterion (5.2), up to the chosen physical representation kernel, and possess the smooth bundle limit of Hypothesis 6.1.*
- (A3) *The record geometry has a smooth four-dimensional time-oriented spin-metric limit with a calibrated conformal factor and a corresponding compact Riemannian spin geometry on which the spectral asymptotics are defined.*
- (A4) *The finite geometry is the Standard Model benchmark representation of (7.2), with the grading, opposite action, allowed transition matrices, and physical fermion prescription described here. This is an explicit finite-geometry input, not a result of (A1)–(A3).*
- (A5) *The product operator is (10.4), its fluctuation satisfies (10.6), and its dynamics is the spectral action (10.7) together with the physical fermion term. The smooth elliptic hypotheses needed for the expansion hold.*
- (A6) *A physical continuation or reconstruction exists giving a Lorentzian theory with the required kinetic signs and physical state space. At the energies considered, omitted higher-order terms and threshold effects are controlled.*

Then the descended transport has the gauge and curvature laws (6.1); the benchmark matter action has the faithful group quotient and anomaly cancellation calculated in Section 8; and the leading spectral densities include gravitational, Yang–Mills, Higgs kinetic and potential, and matter transition terms. Canonical gauge normalization gives (13.8) in the benchmark, with the gravitational coefficient (10.18) in its massless limit. These statements do not determine the finite input, profile, cutoff scale, transition matrices, or quantum continuum measure.

Proof. The behavioral proposition gives well-defined quotient history action. The oriented phase theorem gives complex kinematics on the specified sector. The transport descent theorem gives the retained maps, and the smooth-limit hypothesis converts them into a connection with the local law proved in Section 6. The explicit finite representation supplies the group and matter charges; the anomaly and kernel computations establish the stated benchmark properties. The product and first-order assumptions allow the known real inner-fluctuation construction. The heat-kernel expansion (10.15) gives the local densities, including the explicit gravity and gauge terms derived in Section 10 and the scalar transitions. Equation (13.3) and the trace sums

yield the matching relation. The physical-continuation and truncation assumptions transfer the stipulated effective structure to the Lorentzian regime. Every step uses a stated assumption or a calculation above; no step selects (A4) or (A5) from completion alone. $\square$

The theorem is a conditional assembly statement. Its role is to show exactly what a primitive DTT model would have to produce to use the spectral route consistently. Because its strongest physics inputs include the finite geometry and action, it cannot be cited as a first-principles unification theorem.

14.1 A stronger theorem that remains open

The fundamental target would replace (A3)–(A5) by results. One would begin with a mathematically specified completion law and prove that its stable representation has four-dimensional Lorentzian record geometry, the selected finite algebra and chiral bimodule, a dynamical product operator, and effective action coefficients with reduced freedom. One would also prove that alternative stable algebras and mirror spectra are excluded. No such theorem is established in the present work or implied by the three cited foundations.

14.2 Dependency audit

Object	Sufficient input used here	What remains to be derived
Autonomous record state	Completed-future tests stable under histories	Physically selected tests and completion law
Complex phase	Real orthogonal flow and orientation; kernel treatment	Primitive carrier, flow, and composition rule
Gauge transport	Unitary fibers, links, and descent	Internal fibers, local frame group, and dynamics
Smooth spacetime	Four-dimensional spin limit and calibration	Dimension, smoothness, scale, and metric dynamics
Weak action	Internal doublets separate from spatial spin	Why only left-handed doublets occur
Color action	A retained complex multiplicity of three	Why three colors are stable and selected
Hypercharge ratios	Matter ansatz, Yukawas, and anomaly conditions	Matter ansatz and circle normalization
Higgs	Allowed finite transitions in a chosen chiral module	Transition pattern and stable scalar potential
Einstein term	Geometric Dirac operator and $F_2\Lambda^2a_2$	Microscopic action and Newton scale
Gauge matching	Common profile and representation traces	Profile normalization and thresholds
Particle masses	Vacuum and finite transition singular values	Yukawa spectrum and mixing parameters
Quantum theory	Measure, positivity, renormalization, limits	Interacting continuum and ultraviolet control

15 Consistency of the quantum and continuum interpretation

15.1 Positivity and the Euclidean–Lorentzian interface

The positive Euclidean gauge kinetic density follows from compact internal unitarity and the spectral coefficient. A positive classical density is not reflection positivity of a quantum measure. Osterwalder–Schrader reconstruction relates suitable Euclidean correlation functions to a Lorentzian quantum theory when its detailed axioms hold [23]. A compact heat-kernel calculation, by itself, neither supplies those correlation functions nor proves the axioms. For curved records without a global analytic time coordinate, a naive global Wick rotation may not even be available.

A DTT model therefore needs either a direct Lorentzian formulation with a positive physical state space or a justified Euclidean reconstruction on an appropriate domain. Merely inserting a split-complex unit j with $j^2 = +1$ does not change the number of negative metric directions or prove unitarity. The phase $J_K^2 = -1$ used here is an operator-valued complex structure and should not be confused with that split-complex algebraic unit.

15.2 Causality and local observables

A successful interacting limit must specify its local observable algebras and show compatibility with causal separation. Smooth link convergence on a fixed background does not establish commutation of local quantum observables. Likewise a finite speed bound on a discrete Hamiltonian does not imply exact Poincaré symmetry at all scales. Rotational or Lorentz-violating operators must either vanish in the scaling limit or have constrained coefficients. Their suppression requires a derivation from the dynamics, not just a readout cone.

15.3 Chiral fermions on discrete records

Placing a continuum chiral module at every network vertex gives a bookkeeping model; it does not by itself solve the regularization of chiral fermions. A local, translation-invariant, Hermitian lattice fermion operator with the usual exact chiral symmetry encounters the familiar fermion-doubling obstruction [24]. A completion-based regulator must state which hypothesis is changed or how a known chiral regularization is implemented. Projecting away mirror states without verifying gauge invariance and anomalies is insufficient.

15.4 A temporal spectral gap is not a Yang–Mills mass-gap proof

The circle derivative $-i\partial_\theta/R$ has eigenvalues n/R . Removing its zero mode yields a gap $1/R$ in that operator. It does not prove a positive gap in the physical Hamiltonian of four-dimensional interacting Yang–Mills theory. The latter needs a gauge-invariant continuum construction, a physical vacuum, and a spectral gap above that vacuum. Gauss-law constraints do not generically remove every constant or zero-frequency sector. Compact mode counting, without those steps, should not be represented as a solution of the Yang–Mills existence and mass-gap problem.

15.5 Discrete does not automatically mean ultraviolet complete

At a fixed finite network a finite-dimensional matrix action can be well-defined, but physical limits involve an increasing number of records and possibly unbounded local spectra. Existence of each regulator does not prove existence, uniqueness, positivity, or universality of the continuum limit. Moreover a fundamental cutoff can be physical rather than removable, but then its

symmetry violations and residual scale effects must be predicted and compared with data. The interpretation chosen must be explicit.

16 A constructive research program

16.1 Specify the primitive model before fitting couplings

A concrete next step is a finite completion category or network model with an independently motivated generator set, a composition rule, a completion equivalence, a positive representation, and an action. Its input should be small enough that alternative stable representations can be classified. The generator set should not contain a hidden list of Standard Model particles or matrix sizes chosen from them.

The model should then be tested in the following order: exact descent on a finite network; compatibility of the phase with transport; computation of the residual symmetry group; classification of stable multiplicities and bimodules; anomaly and chirality checks; and only then continuum and coupling calculations. This ordering prevents a numerical fit from disguising a structural assumption.

16.2 A phase–transport compatibility test

If selected phases vary between record fibers, complex-linear transport requires

$$L_{sr}J_r = J_sL_{sr}. \quad (16.1)$$

In a smooth limit this is $\nabla J = 0$. A loop then commutes with the phase at its basepoint. If (16.1) fails, parallel transport does not preserve the reconstructed complex state space. One must account for the changing phase as additional geometric data, restrict transport to its stabilizer, or revise the physical carrier. This is a direct algebraic test, not a matter of interpretation.

16.3 Where DTT could make a distinctive contribution

The completion-to-transport descent criterion is a possible source of constraints absent from a purely prescribed almost-commutative model. A completion law could forbid vertical holonomies visible to retained matter, select compatible left/right actions, or restrict allowed transition blocks. Such results would become substantive if derived from a specified law and shown to exclude competing particle models. They should be evaluated against alternative completion maps and test families, not only the one that reproduces the desired result.

A second possibility is that a primitive stability principle selects spectral weights or a cutoff profile. That could turn relations such as (13.3) into predictions. The principle must constrain both amplitude and shape of the profile; fixing only a matrix algebra leaves F_0 , F_2 , and the absolute scale undetermined. It should also give a calculable error term and a renormalization prescription.

16.4 Falsifiable consequences and stopping criteria

No phenomenological deviation is predicted here because no microscopic parameter set is selected. A completed model could predict a definite matching relation, an additional scalar or sterile spectrum, a pattern of dimension-six operators, or a constrained dispersion correction. Each prediction requires a coefficient and an uncertainty, together with its allowed energy domain. Nonzero higher spectral terms alone do not determine experimental deviations without those coefficients and background solutions.

Failure tests are equally valuable. If a primitive law permits arbitrary color multiplicity, if its physical quotient retains observable vertical holonomy without added state variables, or if phase selection removes every candidate non-Abelian weak action, then that model fails at the corresponding structural stage. If it reproduces the Standard Model only after freely prescribing the entire finite geometry and all its parameters, it remains a common-description model rather than a fundamental derivation.

17 Discussion and conclusion

The central question admits a precise qualified answer. DTT can organize a conditional route to a common effective geometry for gravity and the three gauge sectors. The record viewpoint suggests where spacetime and internal transport could be generated together. The explicit proofs show how compatible phases, descended transports, local gauge laws, and holonomy fit into that viewpoint.

The limits are equally precise. Phase selection does not determine a weak group or three colors. A circle character does not determine the electromagnetic coupling. A causal cone does not determine Einstein dynamics. A generalized connection does not make a curvature-square action equal to the Einstein action. A finite transition matrix does not predict fermion masses. Spectral matching relates coefficients only after a finite module, action profile, and scale have been supplied.

The almost-commutative spectral construction demonstrates that one specified geometric functional can contain gravitational, gauge, Higgs, and matter terms. That established mechanism is a useful conditional destination for DTT, rather than a newly derived DTT result. The outstanding fundamental task is to obtain its decisive data from a primitive completion law, and then to prove physical consistency and predictive control.

Accordingly the present paper establishes a self-contained mathematical architecture, explicit descent and stabilizer constraints, benchmark representation calculations, and a rigorous separation of hypotheses from consequences. It does not establish that DTT already unifies the fundamental interactions. A successful future temporal unification theorem would have to convert the internal selection problem and the action-selection problem into proved statements, and replace arbitrary spectral parameters by independently constrained observables.

A An explicit finite benchmark representation

This appendix makes the finite representation input concrete. It is a benchmark construction from known particle geometry, not an output of a tick law.

A.1 Algebra action and reality

Represent a quaternion as

$$h = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}. \quad (\text{A.1})$$

For one generation with a right-handed neutrino, let the particle space be

$$\mathcal{H}_p = \mathcal{H}_{\ell L} \oplus \mathcal{H}_{q L} \oplus \mathcal{H}_{\nu R} \oplus \mathcal{H}_{e R} \oplus \mathcal{H}_{u R} \oplus \mathcal{H}_{d R}, \quad (\text{A.2})$$

with respective complex dimensions 2, 6, 1, 1, 3, 3. Let $\mathcal{H}_{\bar{p}}$ be a conjugate-labeled copy, and $\mathcal{H}_F = \mathcal{H}_p \oplus \mathcal{H}_{\bar{p}}$, of dimension 32. For $(\lambda, h, m) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, define

$$\pi_p(\lambda, h, m) = h \oplus (h \otimes I_3) \oplus \lambda \oplus \bar{\lambda} \oplus (\lambda I_3) \oplus (\bar{\lambda} I_3), \quad (\text{A.3})$$

$$\pi_{\bar{p}}(\lambda, h, m) = (\lambda I_2) \oplus (I_2 \otimes m) \oplus \lambda \oplus \lambda \oplus m \oplus m. \quad (\text{A.4})$$

These are real-linear unital $*$ -representations. Define

$$\mathcal{J}_F(\psi, \chi) = (\bar{\chi}, \bar{\psi}), \quad \gamma_F = \text{diag}(\gamma_p, -\bar{\gamma}_p), \quad (\text{A.5})$$

where γ_p is +1 on the two left sectors and -1 on the four right sectors. Then $\mathcal{J}_F^2 = 1$ and $\mathcal{J}_F \gamma_F = -\gamma_F \mathcal{J}_F$. The left and opposite actions commute: on quark doublets they act on weak and color factors separately, and on the remaining sectors one of the relevant actions is scalar. Hence the order-zero condition holds.

A.2 An explicit central lift and charge computation

Take $s \in SU(2)$, $C \in SU(3)$, and $\phi \in \mathbb{R}/(2\pi\mathbb{Z})$, and choose algebra unitaries

$$\lambda = e^{3i\phi}, \quad h = s, \quad m = e^{-i\phi} \bar{C}. \quad (\text{A.6})$$

They obey the connected central relation $\lambda \det m = 1$. A determinant condition imposed on the full finite module gives this relation on the identity component; disconnected central possibilities depend on the precise convention. Here (A.6) explicitly supplies the connected lift.

The gauge action on particles is $\pi_p(u) \overline{\pi_{\bar{p}}(u)}$. Its six blocks are therefore

$$\ell_L : e^{-3i\phi} s, \quad q_L : e^{i\phi} (s \otimes C), \quad (\text{A.7})$$

$$\nu_R : 1, \quad e_R : e^{-6i\phi}, \quad (\text{A.8})$$

$$u_R : e^{4i\phi} C, \quad d_R : e^{-2i\phi} C. \quad (\text{A.9})$$

The integer weights are $6Y$. Charge conjugation gives the left-handed table in Section 8. Thus the algebra representation and chosen lift explicitly reproduce the benchmark hypercharges. This computation also identifies the supplied central relation that a fundamental DTT derivation would have to explain.

A.3 Finite transitions and the order-one condition

Group particle states into left and right spaces and use

$$D_p = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \quad M = M_\ell \oplus (M_q \otimes I_3), \quad (\text{A.10})$$

where

$$M_\ell = \text{diag}(m_\nu, m_e), \quad M_q = \text{diag}(m_u, m_d). \quad (\text{A.11})$$

Let $D_F = D_p \oplus \bar{D}_p$, optionally with a self-adjoint Majorana link between ν_R and ν_R^c . The conjugate link is chosen so that $\mathcal{J}_F D_F = D_F \mathcal{J}_F$. All links connect opposite grading sectors, so $\gamma_F D_F = -D_F \gamma_F$.

For the Dirac transition blocks, the opposite action is the same scalar on both lepton endpoints, and the same color matrix on both quark endpoints. Thus $[D_F, \pi(a)]$ commutes with the opposite action, proving the order-one condition for these blocks. On the sterile Majorana link the left action is the same scalar at both endpoints in (A.4); its commutator is zero. This checks the stated order-one condition there as well. Stronger optional spectral axioms, such as

a particular orientability condition in the sterile sector, require a separate treatment and are not asserted here.

With several generations, m_f becomes a matrix on generation space and the algebra acts identically on that factor. The same commutator checks hold. Those matrices remain arbitrary subject to self-adjointness, reality, and any separately imposed flavor constraints.

A.4 A Higgs transition matrix

The quaternion-valued transition fluctuation can be written

$$\Phi(H) = \begin{pmatrix} \bar{H}_2 & H_1 \\ -\bar{H}_1 & H_2 \end{pmatrix} = (\tilde{H} \ H). \quad (\text{A.12})$$

Its first column carries the conjugate hypercharge of its second column. Multiplication of the appropriate M_ℓ, M_q blocks by this matrix supplies the neutrino/up and charged-lepton/down transitions. Under (A.6), the second column transforms as $H \mapsto e^{3i\phi} s H$, exactly the integer weight $6Y = 3$. This verifies the required scalar representation. The fact that the usual finite one-form construction selects this quaternionic transition pattern is part of the benchmark finite geometry; a general off-diagonal matrix would not be restricted to a single Higgs doublet.

B Network action and its continuum expansion

B.1 A positive Wilson-type functional

For an $SU(N)$ representation on an oriented hypercubic cell complex of spacing ϵ , define a plaquette transport U_p and

$$S_\epsilon = C \sum_p (N - \Re \text{tr} U_p), \quad C > 0. \quad (\text{B.1})$$

It is gauge invariant by conjugation covariance and nonnegative by unitarity. This is the standard Wilson-type regulator [22], interpreted here on retained record links. It presupposes a cell structure, an allowed group, and a choice of weights; it is not selected by completion equivalence alone.

For links sampled from a smooth connection,

$$U_p = I \pm \epsilon^2 \mathcal{F}_{\mu\nu} + O(\epsilon^3), \quad (\text{B.2})$$

and the real trace has the expansion

$$N - \Re \text{tr} U_p = -\frac{\epsilon^4}{2} \text{tr}(\mathcal{F}_{\mu\nu}^2) + O(\epsilon^5). \quad (\text{B.3})$$

The coefficient can be checked using the logarithm of the unitary plaquette: its leading anti-Hermitian term is $\pm \epsilon^2 \mathcal{F}_{\mu\nu}$, while its linear real trace vanishes. For each unordered pair $\mu < \nu$ and a fixed smooth compact background, the Riemann sum gives

$$S_\epsilon \longrightarrow -\frac{C}{4} \int \text{tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}) d^4x. \quad (\text{B.4})$$

Choosing $C = 2/g^2$ yields the conventional $-\frac{1}{2g^2} \text{tr} \mathcal{F}^2$ normalization with $\text{tr} T_a T_b = \delta_{ab}/2$. This limit is for classical smooth sampled fields. It is not a convergence theorem for fluctuating quantum measures, singular field configurations, or an emergent metric.

B.2 An exact weak-sector commutator example

Let $A = -ia\sigma_1/2$, $B = -ib\sigma_2/2$, and use the loop

$$U_{\square} = e^{\epsilon A} e^{\epsilon B} e^{-\epsilon A} e^{-\epsilon B}. \quad (\text{B.5})$$

The Baker–Campbell–Hausdorff expansion gives

$$U_{\square} = I + \epsilon^2[A, B] + O(\epsilon^3), \quad [A, B] = -\frac{iab}{2}\sigma_3. \quad (\text{B.6})$$

Its exact trace is

$$\text{tr } U_{\square} = 2 - 4 \sin^2(a\epsilon/2) \sin^2(b\epsilon/2) = 2 - \frac{a^2 b^2 \epsilon^4}{4} + O(\epsilon^6). \quad (\text{B.7})$$

Thus noncommuting transports create a gauge-invariant plaquette discrepancy at fourth order in this trace observable. The nonlinear term is an algebraic consequence of the chosen $SU(2)$ action; the choice of that action and its coupling is still input.

C A small model and its underdetermination

Take each record fiber to be the realification of $\mathbb{C}^n$, choose $K_r = \omega J$ with the same $\omega > 0$, and choose all links in $U(n)$. Then (16.1) holds exactly, selected phase is preserved, network holonomy is well-defined, and the smooth limit, when supplied, gives a $U(n)$ connection with curvature $\mathcal{F}$.

For $n = 1$ this is an Abelian transport model. For $n = 2$, a determinant restriction gives an $SU(2)$ model. For $n = 3$, it gives an $SU(3)$ model. For $n = 4$, it gives an equally valid $SU(4)$ model under the same phase and descent assumptions. Any positive coefficient C can multiply its Wilson-type action without changing the selected phase or internal representation. If all fibers are projected into a one-dimensional record that does not retain nontrivial holonomy, descent fails unless the links act trivially on the retained sector.

This example is deliberately simple. It proves that the transport mechanism works and that its structural and numerical freedom remains. A successful unification proposal must add a principle that excludes the unwanted values of n and fixes or constrains C . Simplicity, ontological unity, and discreteness are not by themselves that principle.

D Notation and dimensional audit

Symbol	Meaning	Unit or mathematical role
K	Skew-adjoint generative-flow generator	inverse generative time
J_K	Selected real-linear phase	dimensionless; $J_K^2 = -1$
$\mathcal{J}_F$	Spectral real structure	antiunitary, not J_K
D_F	Finite transition Dirac operator	inverse length in spectral action
ω_j	A stipulated physical rest frequency	inverse physical time
$\mathcal{B}_\mu$	Anti-Hermitian gauge connection	inverse length
$\mathcal{F}_{\mu\nu}$	Its curvature	inverse length squared
Λ	Spectral cutoff used here	inverse length
F_0, F_2, F_4	Profile moments with stated argument	dimensionless
K_i	Representation-trace normalization	dimensionless
g_i	Canonically normalized four-dimensional gauge coupling	dimensionless
G	Newton constant in natural units	length squared

The algebraic form of a relation survives a unit convention, but a physical derivation must identify the calibration that makes its entries measured quantities. In particular, the symbols K , D_F , and ω_j cannot be equated merely because they all have spectra. An intertwining relation and a conversion scale are required.

References

- [1] M. Haj Yousef, Completion principles, real commutants, and positive-frequency geometry for orthogonal Hilbert representations, *Open Journal of Mathematical Sciences* **10** (2026), 996–1012. doi:10.30538/oms2026.0328.
- [2] M. Haj Yousef, Real-commutant phase selection from real Hilbert representations, *Physics Letters A* **594** (2026), 132085. doi:10.1016/j.physleta.2026.132085.
- [3] M. Haj Yousef, Dual-Time Topological Geometry and the Emergence of Temporal Asymmetry in Non-Equilibrium Dynamics, *Mathematics* **14**(5) (2026), 853. doi:10.3390/math14050853.
- [4] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis*, revised and enlarged edition, Academic Press, 1980.
- [5] J. B. Conway, *A Course in Functional Analysis*, second edition, Springer, 1990.
- [6] V. Moretti and M. Oppio, Quantum theory in real Hilbert space: How the complex Hilbert space structure emerges from Poincaré symmetry, *Reviews in Mathematical Physics* **29** (2017), 1750021. doi:10.1142/S0129055X17500210.
- [7] I. E. Segal, Postulates for general quantum mechanics, *Annals of Mathematics* **48** (1947), 930–948.
- [8] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.
- [9] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, 2001.

- [10] A. H. Chamseddine and A. Connes, The spectral action principle, *Communications in Mathematical Physics* **186** (1997), 731–750. doi:10.1007/s002200050126.
- [11] A. H. Chamseddine, A. Connes and M. Marcolli, Gravity and the standard model with neutrino mixing, *Advances in Theoretical and Mathematical Physics* **11** (2007), 991–1089. doi:10.4310/ATMP.2007.v11.n6.a3.
- [12] A. H. Chamseddine and A. Connes, Conceptual explanation for the algebra in the non-commutative approach to the standard model, *Physical Review Letters* **99** (2007), 191601. doi:10.1103/PhysRevLett.99.191601.
- [13] D. V. Vassilevich, Heat kernel expansion: user’s manual, *Physics Reports* **388** (2003), 279–360. doi:10.1016/j.physrep.2003.09.002.
- [14] J. C. Baez, The octonions, *Bulletin of the American Mathematical Society* **39** (2002), 145–205. doi:10.1090/S0273-0979-01-00934-X.
- [15] H. Mori, Transport, collective motion, and Brownian motion, *Progress of Theoretical Physics* **33** (1965), 423–455.
- [16] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, Oxford University Press, 2002.
- [17] E. H. Lieb and D. W. Robinson, The finite group velocity of quantum spin systems, *Communications in Mathematical Physics* **28** (1972), 251–257.
- [18] M. B. Hastings and T. Koma, Spectral gap and exponential decay of correlations, *Communications in Mathematical Physics* **265** (2006), 781–804.
- [19] D. B. Malament, The class of continuous timelike curves determines the topology of spacetime, *Journal of Mathematical Physics* **18** (1977), 1399–1404. doi:10.1063/1.523436.
- [20] D. Lovelock, The Einstein tensor and its generalizations, *Journal of Mathematical Physics* **12** (1971), 498–501. doi:10.1063/1.1665613.
- [21] E. Witten, An $SU(2)$ anomaly, *Physics Letters B* **117** (1982), 324–328. doi:10.1016/0370-2693(82)90728-6.
- [22] K. G. Wilson, Confinement of quarks, *Physical Review D* **10** (1974), 2445–2459. doi:10.1103/PhysRevD.10.2445.
- [23] K. Osterwalder and R. Schrader, Axioms for Euclidean Green’s functions, *Communications in Mathematical Physics* **31** (1973), 83–112. doi:10.1007/BF01645738.
- [24] H. B. Nielsen and M. Ninomiya, Absence of neutrinos on a lattice: I. Proof by homotopy theory, *Nuclear Physics B* **185** (1981), 20–40; erratum **195** (1982), 541.