

Vacuum Obstructions, Protected Band Geometry, and Quadratic-Form Feshbach Reduction in Auxiliary-Time Lattice Gauge Hamiltonians

Exact finite-regulator dynamics, holonomy-twisted interruption, and a punctured low-window criterion

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Abstract

We construct an exact finite-regulator auxiliary-time lattice gauge Hamiltonian and analyze a natural low/high decomposition defined by mixed spatial/auxiliary plaquette energy. The positive semigroup has a unique strictly positive vacuum and a reversible two-sided Osterwalder–Schrader process. Mixed plaquettes are exact Hilbert–Schmidt squares of covariant auxiliary differences; axial gauge yields holonomy-twisted discrete spectra and shows that compactness of the auxiliary cycle alone provides no uniform mass floor.

The principal result is a vacuum obstruction. For the vacuum-corrected transfer-gap operator, any nonconstant multiplication window that has nonzero vacuum weight on both sides makes the natural unpunctured relative-mixing inequality fail strictly on the transported vacuum component. We prove the exact identity responsible for the failure, construct the transported vacuum line, and establish a corrected punctured Schur–Feshbach theorem with an explicit physical-time-step margin. We also give a quadratic-form Feshbach identity appropriate for unbounded gauge Schrödinger operators.

Finally, we introduce a gauge-invariant auxiliary-holonomy closure term and an exact Friedrichs completion Hamiltonian. For one frozen diagonal scaling, the linearized relative and holonomy stiffnesses diverge while the bare local closure coefficient vanishes. We also record the intrinsic Kato connection of any smooth isolated completion band, correct the common but vanishing expression $P dP|_{\text{Ran } P}$, and formulate the no-extra-gauge condition required when the band is an auxiliary regulator rather than a retained physical carrier. These are finite-regulator, linearized, and geometric results; they do not prove a nonlinear completion band, a regulator-uniform four-dimensional mass gap, a continuum Yang–Mills theory, or universality. We conclude with a precise fast-sector problem for $SU(2)$ whose solution would test whether the auxiliary completion construction provides new analytic leverage.

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1 Introduction and main results

The purpose of this article is to isolate a technically complete part of an auxiliary-time lattice gauge program. Ordinary Euclidean time is the transfer and reflection direction. A finite cyclic variable τ lies inside each transfer slice and is treated as an auxiliary completion coordinate. The analysis below is independent of any ontological interpretation assigned to that coordinate.

The central issue is a low/high split by a nonnegative multiplication operator K . If the transfer operator T has a strictly positive vacuum Ω , a nonconstant multiplication window generally divides Ω between its two ranges. The vacuum-corrected operator

$$G = rI - T + (1 - r)|\Omega\rangle\langle\Omega|, \quad 0 < r < 1,$$

then has off-diagonal blocks that differ from those of $-T$ by a rank-one term. The following theorem is the main finite-dimensional obstruction.

Theorem 1.1 (Vacuum obstruction for an unpunctured window). *Let T be a bounded positive self-adjoint operator, let Ω be a unit vector with $T\Omega = \Omega$, let $0 < r < 1$, and set*

$$G = rI - T + (1 - r)|\Omega\rangle\langle\Omega|.$$

Let P be an orthogonal projection, $Q = I - P$, and assume

$$\omega = P\Omega \neq 0, \quad \zeta = Q\Omega \neq 0, \quad D = QGQ \geq dQ \quad (d > 0).$$

Then

$$\langle \omega, PTQD^{-1}QTP\omega \rangle > \langle \omega, PGP\omega \rangle > 0. \quad (1)$$

Consequently no $\eta \leq 1$ can satisfy

$$PTQD^{-1}QTP \leq \eta PGP \quad (2)$$

on the whole low space $P\mathcal{H}$.

The proof is given in [section 6.2](#). The obstruction is removed by transporting the vacuum through the high block and restricting the Feshbach operator to the orthogonal complement of $P\Omega$. Under explicit low, high, and mixing estimates, [theorem 6.8](#) gives a positive margin of order a_t , which is the correct scaling for a fixed physical Hamiltonian gap.

The other completed components are the exact finite-regulator Hamiltonian and vacuum process, the holonomy-twisted spectral calculation, an exact completion Hamiltonian with a holonomy closure potential, a frozen linear fast-scale window, the Kato geometry of a smooth isolated band, and the quadratic-form Feshbach identity in [theorem 10.3](#). None of these results is presented as a proof of four-dimensional Yang–Mills existence or a regulator-uniform mass gap.

2 Gauge group, defect function, and frozen scales

2.1 Compact simple-Lie-algebra gauge groups

Fix throughout a compact connected Lie group G whose Lie algebra $\mathfrak{g}$ is simple. The global form of G is not fixed: it may be simply connected, adjoint, or an intermediate quotient. Let

$$\pi : \tilde{G} \longrightarrow G, \quad \Gamma = \text{Ker } \pi \cong \pi_1(G),$$

be the universal covering homomorphism. Then Γ is finite and central. Fix an Ad-invariant inner product $(\cdot, \cdot)_{\mathfrak{g}}$, normalized so that the adjoint quadratic Casimir is $C_A > 0$. Fix once and for all a faithful finite-dimensional unitary representation

$$\rho : G \longrightarrow U(d_\rho).$$

There is a unique $c_\rho > 0$ such that

$$-\text{Tr}(d\rho(X)d\rho(Y)) = c_\rho(X, Y)_{\mathfrak{g}}.$$

Define the normalized plaquette defect

$$\ell_\rho(g) = c_\rho^{-1} \text{Re Tr}(I_{d_\rho} - \rho(g)). \quad (3)$$

Lemma 2.1 (Plaquette-defect normalization). *The function ℓ_ρ is continuous, nonnegative, conjugation invariant, and satisfies*

$$\ell_\rho(g) = 0 \iff g = e.$$

For $X \rightarrow 0$ in $\mathfrak{g}$,

$$\ell_\rho(\exp X) = \frac{1}{2}(X, X)_\mathfrak{g} + O(\|X\|^4).$$

Proof. Every eigenvalue of the unitary matrix $\rho(g)$ has the form $e^{i\theta_j}$, so

$$\operatorname{Re} \operatorname{Tr}(I - \rho(g)) = \sum_j (1 - \cos \theta_j) \geq 0.$$

It vanishes exactly when every eigenvalue equals one, hence when $\rho(g) = I$. Faithfulness gives $g = e$. Conjugation invariance and continuity follow from the trace. Since $\mathfrak{g}$ is semisimple, $\operatorname{Tr}(d\rho(X)) = 0$. Expanding the exponential and taking real parts gives

$$\operatorname{Re} \operatorname{Tr}(I - e^{d\rho(X)}) = -\frac{1}{2} \operatorname{Tr}(d\rho(X)^2) + O(\|X\|^4) = \frac{c_\rho}{2}(X, X)_\mathfrak{g} + O(\|X\|^4).$$

□

2.2 Physical scale and the diagonal regulator sequence

Fix a reference inverse length $\Lambda_* > 0$. All dimensionful constants are measured relative to Λ_* ; in particular a proposed physical mass m means m/Λ_* is held fixed as the regulator is removed. For integers $n \geq n_0$, define

$$\lambda_n = (a_n, a_{t,n}, L_n, a_{\tau,n}, N_{\tau,n}) = \left(\frac{2^{-n}}{\Lambda_*}, \frac{2^{-n}}{\Lambda_*}, 2^{2n}, \frac{2^{-2n}}{\Lambda_*}, 2^n \right). \quad (4)$$

Here L_n is the number of sites in each spatial direction. Thus

$$a_n \Lambda_* = a_{t,n} \Lambda_* = 2^{-n}, \quad a_{\tau,n} \Lambda_* = 2^{-2n}, \quad (5)$$

$$R_n := L_n a_n = \frac{2^n}{\Lambda_*} \rightarrow \infty, \quad \beta_{\tau,n} := N_{\tau,n} a_{\tau,n} = \frac{2^{-n}}{\Lambda_*} \rightarrow 0.$$

This is one diagonal family, not a collection of unspecified iterated limits.

2.3 A precise two-loop reference bare-coupling sequence

Set

$$b_0 = \frac{11C_A}{3(4\pi)^2}, \quad b_1 = \frac{34C_A^2}{3(4\pi)^4}. \quad (6)$$

For $x > 0$, define

$$\Phi_G(x) = \exp\left(-\frac{1}{2b_0x}\right) (b_0x)^{-b_1/(2b_0^2)}. \quad (7)$$

Choose n_0 large enough that $2^{-n} < \Phi_G(b_0/(2b_1))$ for every $n \geq n_0$, and define $g_n > 0$ to be the unique solution in

$$0 < g_n^2 < \frac{b_0}{2b_1}$$

of

$$a_n \Lambda_* = 2^{-n} = \Phi_G(g_n^2). \quad (8)$$

Proposition 2.2 (Existence and ultraviolet behavior of the reference sequence). *The sequence g_n in equation (8) is well defined, satisfies $g_n \rightarrow 0$, and obeys*

$$g_n^2 = \frac{1}{2b_0 n \log 2} \left(1 + O\left(\frac{\log n}{n}\right) \right). \quad (9)$$

Proof. A direct differentiation gives

$$\frac{d}{dx} \log \Phi_G(x) = \frac{b_0 - b_1 x}{2b_0^2 x^2}.$$

Hence Φ_G is strictly increasing on $(0, b_0/b_1)$, and $\Phi_G(x) \rightarrow 0$ as $x \downarrow 0$. The choice of n_0 gives existence and uniqueness in the declared interval. Taking logarithms of equation (8) yields

$$n \log 2 = \frac{1}{2b_0 g_n^2} + \frac{b_1}{2b_0^2} \log(b_0 g_n^2).$$

The first term forces $g_n \rightarrow 0$. Substitution of the leading estimate back into the logarithmic term gives equation (9). $\square$

Warning 2.3 (What theorem 2.2 does not prove). The relation equation (8) freezes a numerical sequence modeled on the universal perturbative coefficients [4–7]. It does not prove that the nonperturbative lattice measures converge, that the resulting continuum theory is asymptotically free, or that Λ_* equals a dynamically generated mass scale. Those would be separate theorems.

2.4 Frozen auxiliary fast-sector scaling

The completion-band regulator must be frozen before any spectral estimate is attempted. Fix constants

$$0 < \sigma_H < 1, \quad \kappa_* > 0,$$

and set

$$\kappa_{H,n} = \kappa_* \Lambda_* N_{\tau,n}^{-(1-\sigma_H)}, \quad c_{\tau,n}^{\text{kin}} = \frac{g_n^2 a_n}{2a_{\tau,n}^2 N_{\tau,n}}, \quad \alpha_n^{\text{mix}} = \frac{a_n}{2g_n^2 N_{\tau,n}}. \quad (10)$$

The coefficient $c_{\tau,n}^{\text{kin}}$ is the exact auxiliary-link Laplace coefficient inherited from $\mathcal{L}_n$. The coefficient α_n^{mix} is the tangent coefficient of the normalized mixed-curvature potential after the expansion $\ell_\rho(e^X) = \frac{1}{2}\|X\|^2 + O(\|X\|^4)$. The closure coefficient tends to zero as a bare local coupling, while its coherent mean-mode Hessian is amplified by the product around the $N_{\tau,n}$ -site auxiliary cycle.

Proposition 2.4 (Algebraic fast-scale window). *For the diagonal and coupling sequences in (4) and (8),*

$$c_{\tau,n}^{\text{kin}} = \frac{g_n^2}{2} 2^{2n} \Lambda_*, \quad (11)$$

$$\gamma_{\text{rel},n}^{\text{lin}} := \frac{16\alpha_n^{\text{mix}}}{\beta_{\tau,n}^2} = \frac{8\Lambda_*}{g_n^2}, \quad (12)$$

$$\gamma_{H,n}^{\text{lin}} := \frac{1}{2} \kappa_{H,n} N_{\tau,n} = \frac{\kappa_* \Lambda_*}{2} N_{\tau,n}^{\sigma_H}. \quad (13)$$

Consequently

$$c_{\tau,n}^{\text{kin}} \longrightarrow \infty, \quad \gamma_{\text{rel},n}^{\text{lin}} \longrightarrow \infty, \quad \gamma_{H,n}^{\text{lin}} \longrightarrow \infty,$$

whereas

$$a_{t,n}\kappa_{H,n} = \kappa_* 2^{-n(2-\sigma_H)} \longrightarrow 0. \quad (14)$$

Thus the frozen sequence has a nonempty linear scaling window in which the auxiliary tangent stiffness diverges but the one-step bare closure coefficient vanishes.

Proof. Insert $a_n = a_{t,n} = 2^{-n}\Lambda_*^{-1}$, $a_{\tau,n} = 2^{-2n}\Lambda_*^{-1}$, $N_{\tau,n} = 2^n$, and $\beta_{\tau,n} = a_n$ into [equation \(10\)](#). Since $N_{\tau,n}a_n = \Lambda_*^{-1}$,

$$\frac{16\alpha_n^{\text{mix}}}{\beta_{\tau,n}^2} = \frac{8}{g_n^2 N_{\tau,n} a_n} = \frac{8\Lambda_*}{g_n^2}.$$

The other identities are immediate. Finally $g_n \rightarrow 0$ by [theorem 2.2](#), so all three positive fast scales diverge. The limit (14) follows directly from the chosen exponent $0 < \sigma_H < 1$. $\square$

Remark 2.5 (Scope of the fast-scale calculation). The divergence of the linear coefficients is a tangent-space statement. It does not prove a nonlinear spectral band, uniform gauge-chart control, absence of remote wells, or smallness of the induced four-dimensional interactions.

3 Finite-regulator heat-kernel transfer theory

3.1 Slice lattice, variables, and gauge action

For fixed n , let

$$\Sigma_n = (\mathbb{Z}/L_n\mathbb{Z})^3, \quad C_n = \mathbb{Z}/N_{\tau,n}\mathbb{Z}.$$

A vertex of one outer-time slice is $(x, \tau) \in \Sigma_n \times C_n$. There are spatial links

$$U_i(x, \tau) \in G, \quad i = 1, 2, 3,$$

and auxiliary links

$$V(x, \tau) \in G.$$

The untwisted slice configuration space is the compact product

$$\mathcal{X}_n = G^{3L_n^3 N_{\tau,n}} \times G^{L_n^3 N_{\tau,n}}, \quad (15)$$

with normalized product Haar measure μ_n . The slice gauge group

$$\mathcal{G}_n = G^{L_n^3 N_{\tau,n}}$$

acts by

$$U_i^g(x, \tau) = g(x, \tau) U_i(x, \tau) g(x + \hat{i}, \tau)^{-1}, \quad (16)$$

$$V^g(x, \tau) = g(x, \tau) V(x, \tau) g(x, \tau + 1)^{-1}. \quad (17)$$

Indices are periodic.

The spatial and mixed plaquettes are

$$W_{ij}(x, \tau) := U_i(x, \tau) U_j(x + \hat{i}, \tau) U_i(x + \hat{j}, \tau)^{-1} U_j(x, \tau)^{-1}, \quad (18)$$

$$W_{i\tau}(x, \tau) := V(x, \tau) U_i(x, \tau + 1) V(x + \hat{i}, \tau)^{-1} U_i(x, \tau)^{-1}. \quad (19)$$

Both transform by conjugation at (x, τ) .

3.2 The frozen kinetic and potential coefficients

Let Δ_ℓ denote the nonnegative Laplace–Beltrami operator on the copy of G attached to link ℓ , for the bi-invariant metric induced by $(\cdot, \cdot)_g$. Define on $C^\infty(\mathcal{X}_n)$

$$\mathcal{L}_n = \frac{g_n^2}{2a_n N_{\tau,n}} \sum_{x,\tau,i} \Delta_{U_i(x,\tau)} + \frac{g_n^2 a_n}{2a_{\tau,n}^2 N_{\tau,n}} \sum_{x,\tau} \Delta_{V(x,\tau)}. \quad (20)$$

The within-slice potential is

$$\begin{aligned} \mathcal{V}_n(U, V) &= \frac{1}{g_n^2 a_n N_{\tau,n}} \sum_{x,\tau} \sum_{1 \leq i < j \leq 3} \ell_\rho(W_{ij}(x, \tau)) \\ &\quad + \frac{a_n}{g_n^2 a_{\tau,n}^2 N_{\tau,n}} \sum_{x,\tau,i} \ell_\rho(W_{i\tau}(x, \tau)). \end{aligned} \quad (21)$$

The mixed-energy function and multiplication operator are

$$\mathcal{K}_n(U, V) := \frac{a_n}{a_{\tau,n}^2 N_{\tau,n}} \sum_{x,\tau,i} \ell_\rho(W_{i\tau}(x, \tau)), \quad (22)$$

$$(K_n f)(U, V) := \mathcal{K}_n(U, V) f(U, V). \quad (23)$$

Thus the mixed part of $\mathcal{V}_n$ is $g_n^{-2} \mathcal{K}_n$. All coefficients are now fixed functions of n ; no adjustable anisotropy remains.

Proposition 3.1 (Basic operator properties). *The operator $\mathcal{L}_n$ is essentially self-adjoint and nonnegative on $C^\infty(\mathcal{X}_n)$, has compact resolvent, and commutes with the gauge action, spatial translations, cubic rotations, and cyclic auxiliary translations. The functions $\mathcal{V}_n$ and $\mathcal{K}_n$ are continuous, bounded, nonnegative, gauge invariant, and invariant under the same lattice symmetries. Consequently K_n is bounded, nonnegative, and self-adjoint.*

Proof. The sum in [equation \(20\)](#) is a strictly elliptic Laplace-type operator on the compact product manifold $\mathcal{X}_n$. Standard elliptic theory gives essential self-adjointness on smooth functions, nonnegativity, and compact resolvent. Bi-invariance of each link metric makes the Laplacian commute with left and right translations, hence with the gauge action. Translation-invariance of the coefficients gives the lattice symmetries. The remaining claims follow from [theorem 2.1](#), compactness of $\mathcal{X}_n$, and conjugation covariance of plaquette holonomies. $\square$

3.3 Gauge-invariant Hilbert space and heat kernel

Set

$$\mathcal{H}_n^{\text{all}} = L^2(\mathcal{X}_n, \mu_n).$$

The gauge-invariant projection and slice Hilbert space are

$$(\Pi_{\mathcal{G},n} f)(X) = \int_{\mathcal{G}_n} f(g^{-1} \cdot X) \, dg, \quad \mathcal{H}_n^{\text{sl}} = \text{Ran } \Pi_{\mathcal{G},n}. \quad (24)$$

The heat semigroup on the unreduced space is

$$\mathbf{P}_n^{\text{all}} = \exp(-a_{t,n} \mathcal{L}_n).$$

Since $\mathcal{L}_n$ commutes with $\Pi_{\mathcal{G},n}$, define

$$\mathbf{P}_n = \mathbf{P}_n^{\text{all}}|_{\mathcal{H}_n^{\text{sl}}}. \quad (25)$$

For a spatial link set

$$s_{s,n} = a_{t,n} \frac{g_n^2}{2a_n N_{\tau,n}},$$

and for an auxiliary link set

$$s_{\tau,n} = a_{t,n} \frac{g_n^2 a_n}{2a_{\tau,n}^2 N_{\tau,n}}.$$

Let h_s be the central heat kernel on G for $\exp(-s\Delta)$. If E_n denotes all spatial and auxiliary links in the slice and $s_{\ell,n}$ is the appropriate one of these two times, the unreduced kernel is

$$\mathbf{p}_n^0(X, Y) = \prod_{\ell \in E_n} h_{s_{\ell,n}}(X_\ell Y_\ell^{-1}). \quad (26)$$

Proposition 3.2 (Gauge-averaged temporal-link kernel). *The restriction $\mathbf{P}_n$ has the separately gauge-invariant smooth kernel*

$$\mathbf{p}_n^G(X, Y) = \int_{\mathcal{G}_n} \prod_{\ell \in E_n} h_{s_{\ell,n}}(X_\ell g_{t(\ell)} Y_\ell^{-1} g_{s(\ell)}^{-1}) \prod_v dg_v. \quad (27)$$

The integration variables g_v can be read as outer-time links between two adjacent slices, and every factor is the heat-kernel weight of the corresponding outer-time plaquette. The kernel is strictly positive.

Proof. The product formula [equation \(26\)](#) follows from commutativity of the link Laplacians. Gauge averaging in the second variable gives

$$\int_{\mathcal{G}_n} \mathbf{p}_n^0(X, g \cdot Y) dg.$$

For a link $s(\ell) \rightarrow t(\ell)$,

$$(g \cdot Y)_\ell = g_{s(\ell)} Y_\ell g_{t(\ell)}^{-1},$$

which gives [equation \(27\)](#). Each compact-group heat kernel is smooth and strictly positive for positive time, and averaging preserves strict positivity. Centrality gives separate gauge invariance and the plaquette interpretation. $\square$

3.4 Transfer operator, vacuum, and Hamiltonian

Let

$$M_n = \exp(-a_{t,n} \mathcal{V}_n / 2) \quad (28)$$

act by multiplication. Since $\mathcal{V}_n$ is bounded, M_n and M_n^{-1} are bounded. Define

$$T_n^{(0)} = M_n \mathbf{P}_n M_n, \quad T_n = \frac{T_n^{(0)}}{\|T_n^{(0)}\|}. \quad (29)$$

Theorem 3.3 (Finite-regulator transfer operator). *On $\mathcal{H}_n^{\text{sl}}$, the operator T_n is self-adjoint, positive as an operator, positivity preserving, injective, trace class, and a contraction. It is positivity improving on the cone of nonnegative gauge-invariant functions. Its spectral radius is the simple eigenvalue 1, with a unique normalized eigenfunction Ω_n that is smooth and strictly positive at every point of $\mathcal{X}_n$.*

Proof. Self-adjointness and operator positivity follow from

$$\langle f, T_n^{(0)} f \rangle = \langle M_n f, P_n M_n f \rangle \geq 0.$$

The heat semigroup is trace class on the compact product and restriction to a reducing subspace preserves that property; multiplication by bounded operators preserves trace class. If $T_n^{(0)} f = 0$, then the displayed quadratic form is zero. Spectral calculus for $\mathcal{L}_n$ gives $\text{Ker } P_n = \{0\}$ and $\langle h, P_n h \rangle > 0$ for every $h \neq 0$, so $M_n f = 0$, and invertibility of M_n yields $f = 0$. The kernel from [theorem 3.2](#), multiplied by the two strictly positive potential factors, is strictly positive. Therefore a nonzero nonnegative function is mapped to a pointwise positive function. The Jentzsch–Perron theorem for compact positivity-improving integral operators gives a simple strictly positive top eigenvector; this is the transfer-matrix mechanism used in rigorous lattice gauge theory [\[10, 11\]](#). Division by the norm makes its eigenvalue one. Smoothness follows from heat-kernel smoothing, and strict positivity is pointwise. Compactness of $\mathcal{X}_n$ also gives $\min \Omega_n > 0$. $\square$

Definition 3.4 (Physical Hamiltonian and domains). Let $E_n(\cdot)$ be the spectral measure of T_n . Define

$$H_n = -a_{t,n}^{-1} \log T_n \quad (30)$$

with

$$\text{Dom}(H_n) = \left\{ f \in \mathcal{H}_n^{\text{sl}} : \int_{(0,1]} a_{t,n}^{-2} |\log t|^2 \, d\langle f, E_n(t) f \rangle < \infty \right\}. \quad (31)$$

Its closed quadratic form q_n has domain

$$\text{Dom}(q_n) = \left\{ f : \int_{(0,1]} a_{t,n}^{-1} |\log t| \, d\langle f, E_n(t) f \rangle < \infty \right\}. \quad (32)$$

Proposition 3.5 (Self-adjointness and fixed-regulator gap). *The operator H_n is densely defined, nonnegative, and self-adjoint; $H_n \Omega_n = 0$. At each fixed n there is a number $\Delta_n > 0$ such that*

$$H_n \geq \Delta_n I \quad \text{on } \Omega_n^\perp. \quad (33)$$

No lower bound on $\inf_n \Delta_n$ follows from this proposition.

Proof. Injectivity of T_n means its spectral projection at zero vanishes. The measurable functional calculus therefore defines $-\log T_n$ on the dense domain [equation \(31\)](#); it is self-adjoint and nonnegative because $0 < T_n \leq I$. Since $T_n \Omega_n = \Omega_n$, the vacuum has energy zero. Compactness and simplicity of the top eigenvalue imply that the remainder of $\text{Spec}(T_n)$ lies in $[0, r_n]$ for some $r_n < 1$. Thus $\Delta_n = -a_{t,n}^{-1} \log r_n > 0$. The argument contains no estimate uniform in n . $\square$

3.5 Physical normalization on the zero-mixed stratum

Suppose $V(x, \tau) = I$ and $U_i(x, \tau) = U_i(x)$ are independent of τ . Then all mixed plaquettes are the identity and

$$\mathcal{V}_n(U, I) = \frac{1}{g_n^2 a_n} \sum_x \sum_{i < j} \ell_\rho(W_{ij}(x)). \quad (34)$$

Thus the declared $1/N_{\tau,n}$ normalization makes the surviving spatial magnetic term exactly the standard Kogut–Susskind normalization associated with ℓ_ρ , rather than $N_{\tau,n}$ copies of it [\[3\]](#). This identity is a finite-lattice normalization statement only.

4 Vacuum process, auxiliary neutrality, and common-space mixed energy

4.1 Ground-state transform

Define the vacuum probability measure

$$d\nu_n(X) = \Omega_n(X)^2 d\mu_n(X) \quad (35)$$

and the unitary

$$U_{\Omega,n} : L^2(\mathcal{X}_n, \nu_n)^{\mathcal{G}_n} \longrightarrow \mathcal{H}_n^{\text{sl}}, \quad U_{\Omega,n}f = \Omega_n f.$$

The transformed transfer operator

$$\hat{T}_n = U_{\Omega,n}^{-1} T_n U_{\Omega,n} \quad (36)$$

has kernel relative to ν_n

$$\hat{t}_n(X, Y) = \frac{t_n(X, Y)}{\Omega_n(X)\Omega_n(Y)}, \quad (37)$$

where t_n is the integral kernel of T_n .

Proposition 4.1 (Reversible Markov operator). *The operator $\hat{T}_n$ is self-adjoint, positivity improving, and satisfies*

$$\hat{T}_n 1 = 1.$$

Equivalently, $\hat{t}_n$ is a strictly positive symmetric Markov kernel relative to ν_n .

Proof. Unitarity transfers self-adjointness and positivity. The eigenvalue equation $T_n \Omega_n = \Omega_n$ gives $\hat{T}_n 1 = 1$. Substitution into the integral formula gives

$$\int \hat{t}_n(X, Y) d\nu_n(Y) = 1.$$

Symmetry follows from symmetry of t_n , and strict positivity follows from [theorem 3.3](#). $\square$

Let $\mathbb{P}_n^{\text{vac}}$ be the unique two-sided stationary Markov measure on $\mathcal{X}_n^{\mathbb{Z}}$ with one-time law ν_n and transition operator $\hat{T}_n$. Its existence follows from the Kolmogorov extension theorem.

Theorem 4.2 (Finite-regulator reflection positivity and OS reconstruction). *Let θ reflect outer time, $(\theta X)_k = X_{-k}$, and put $\Theta F = \overline{F \circ \theta}$. If F is a bounded cylinder function depending only on times $0, 1, \dots, r$, then*

$$\int (\Theta F) F d\mathbb{P}_n^{\text{vac}} = \int |\mathbb{E}_n^{\text{vac}}[F | X_0]|^2 d\nu_n \geq 0. \quad (38)$$

In the sense of the Osterwalder–Schrader reconstruction [8, 9], the resulting OS Hilbert space is naturally unitarily equivalent to $L^2(\nu_n)^{\mathcal{G}_n}$, hence through $U_{\Omega,n}$ to $\mathcal{H}_n^{\text{sl}}$. One-step OS time translation is $\hat{T}_n$, and its physical generator is unitarily equivalent to H_n .

Proof. Conditioned on X_0 , the past and future of a two-sided Markov chain are independent. Reversibility identifies the conditional law of the reflected past with that of the future. Therefore the conditional expectation of $(\Theta F)F$ given X_0 is the modulus square in [equation \(38\)](#). Quotienting positive-time cylinders by the null space of this form identifies a cylinder with its conditional expectation given X_0 . Every bounded gauge-invariant function of X_0 occurs, and these functions are dense in $L^2(\nu_n)^{\mathcal{G}_n}$, proving the Hilbert-space identification. The Markov property identifies one-step translation with $\hat{T}_n$; [equations \(30\)](#) and [\(36\)](#) then identify its generator. $\square$

Remark 4.3 (Periodic loops are not the vacuum). A finite periodic outer-time loop with weight $\text{Tr}(T_n^M)$ is a trace state. It is not the pure vacuum state at finite M . Since the top eigenvalue of T_n is simple, normalized periodic expectations converge to vacuum expectations as $M \rightarrow \infty$ at fixed n . The infinite two-sided process above constructs the vacuum directly and avoids conflating these states.

4.2 Auxiliary translation and the neutral candidate sector

Let $R_{\tau,n}$ shift every link variable by one unit in the auxiliary coordinate. It is unitary and

$$[R_{\tau,n}, \mathcal{L}_n] = [R_{\tau,n}, \mathcal{V}_n] = [R_{\tau,n}, T_n] = [R_{\tau,n}, K_n] = 0. \quad (39)$$

Define

$$P_{0,n} = \frac{1}{N_{\tau,n}} \sum_{j=0}^{N_{\tau,n}-1} R_{\tau,n}^j, \quad \mathcal{N}_n = P_{0,n} \mathcal{H}_n^{\text{sl}}. \quad (40)$$

Proposition 4.4 (Neutral reducing sector). *The space $\mathcal{N}_n$ reduces T_n , H_n , and K_n . Moreover $\Omega_n \in \mathcal{N}_n$.*

Proof. The commutators in [equation \(39\)](#) imply reduction. Since $R_{\tau,n}\Omega_n$ is another strictly positive normalized eigenvector of T_n at eigenvalue one, simplicity from [theorem 3.3](#) gives $R_{\tau,n}\Omega_n = \Omega_n$. $\square$

The label “physical candidate” is intentionally provisional. Auxiliary-translation neutrality does not imply independence of auxiliary holonomy or of relative auxiliary configurations. Proving that all such observables disappear in a continuum limit is the separate auxiliary-invisibility target, which is not reached here.

4.3 The multiplication spectral window

On $\mathcal{N}_n$, define for $\varepsilon > 0$

$$P_{\varepsilon,n} = \mathbf{1}_{[0,\varepsilon]}(K_n)|_{\mathcal{N}_n}, \quad Q_{\varepsilon,n} = I - P_{\varepsilon,n}. \quad (41)$$

Because K_n is multiplication by $\mathcal{K}_n$, $P_{\varepsilon,n}$ is multiplication by the indicator of the gauge-invariant set

$$\mathcal{A}_{\varepsilon,n} = \{X \in \mathcal{X}_n : \mathcal{K}_n(X) \leq \varepsilon\}.$$

The projection need not preserve $\Omega_n^\perp$, because $K_n\Omega_n$ is generally not proportional to Ω_n . This is the reason for using the vacuum-corrected gap operator in [section 6](#).

5 Holonomy stratification, zero sets, and finite-block diagnostics

5.1 Mixed plaquettes as covariant differences

For a spatial link (x, i) at auxiliary step τ , define

$$\nabla_\tau^{\text{cov}} U_i(x, \tau) = \frac{1}{a_{\tau,n}} \left[\rho(V(x, \tau) U_i(x, \tau + 1)) - \rho(U_i(x, \tau) V(x + \hat{i}, \tau)) \right]. \quad (42)$$

Lemma 5.1 (Exact square identity). *For every mixed plaquette,*

$$\ell_\rho(W_{i\tau}(x, \tau)) = \frac{a_{\tau,n}^2}{2c_\rho} \|\nabla_\tau^{\text{cov}} U_i(x, \tau)\|_{\text{HS}}^2. \quad (43)$$

Proof. Put $A = \rho(V(x, \tau)U_i(x, \tau + 1))$ and $B = \rho(U_i(x, \tau)V(x + \hat{i}, \tau))$. Both are unitary, so

$$\|A - B\|_{\text{HS}}^2 = 2d_\rho - 2\text{Re Tr}(A^*B).$$

By cyclicity of the trace, $\text{Re Tr}(A^*B) = \text{Re Tr}(\rho(W_{i\tau}^{-1})) = \text{Re Tr}(\rho(W_{i\tau}))$. Apply [equation \(3\)](#) and divide by $a_{\tau,n}^2$. $\square$

This pointwise identity does not identify the spectrum of the finite-dimensional difference with the spectrum of the multiplication operator K_n or the Hamiltonian H_n .

5.2 Auxiliary axial gauge and holonomy strata

At each spatial site define the auxiliary holonomy

$$H_x = V(x, 0)V(x, 1) \cdots V(x, N_{\tau,n} - 1). \quad (44)$$

Lemma 5.2 (Auxiliary axial gauge). *Every configuration is gauge equivalent to one satisfying*

$$V(x, \tau) = e \quad (0 \leq \tau < N_{\tau,n} - 1), \quad V(x, N_{\tau,n} - 1) = H_x. \quad (45)$$

Residual gauge transformations are independent of τ and act by

$$U_i(x, \tau) \mapsto h_x U_i(x, \tau) h_{x+\hat{i}}^{-1}, \quad H_x \mapsto h_x H_x h_x^{-1}.$$

Proof. Set $g(x, 0) = e$ and recursively

$$g(x, \tau + 1) = g(x, \tau)V(x, \tau) \quad (0 \leq \tau < N_{\tau,n} - 1).$$

Substitution into [equation \(17\)](#) sets the first $N_{\tau,n} - 1$ auxiliary links to the identity and leaves their ordered product on the closing link. A transformation preserving this gauge must have $h(x, \tau + 1) = h(x, \tau)$, hence is independent of τ . $\square$

Theorem 5.3 (Holonomy stratification of the zero-mixed set). *In auxiliary axial gauge, the conditions*

$$W_{i\tau}(x, \tau) = e \quad \text{for every } x, i, \tau \quad (46)$$

are equivalent to the existence of spatial links $U_i(x)$, independent of τ , and holonomies H_x satisfying

$$H_x U_i(x) = U_i(x) H_{x+\hat{i}}. \quad (47)$$

Modulo residual gauge transformations, the zero set is stratified by the simultaneous conjugacy orbits of the holonomy field $\{H_x\}$ subject to [equation \(47\)](#). The stratum $H_x = e$ for all x is exactly the ordinary three-dimensional spatial lattice-gauge configuration quotient at one outer-time slice.

Proof. For $0 \leq \tau < N_{\tau,n} - 1$, [equations \(19\)](#) and [\(45\)](#) reduce the condition $W_{i\tau} = e$ to

$$U_i(x, \tau + 1) = U_i(x, \tau).$$

Thus U_i is independent of τ . The closing mixed plaquette is

$$H_x U_i(x) H_{x+\hat{i}}^{-1} U_i(x)^{-1},$$

so it equals the identity exactly when [equation \(47\)](#) holds. Reversing these steps proves the converse. The residual action is given in [theorem 5.2](#). If every $H_x = e$, only the τ -independent spatial links and ordinary spatial gauge transformations remain. $\square$

Corollary 5.4 (Ordinary Yang–Mills stratum). *On the trivial-holonomy part of the zero-mixed set, $\mathcal{K}_n = 0$ and the potential reduces to [equation \(34\)](#). Hence every positive low window contains neighborhoods of configurations whose auxiliary reduction is ordinary lattice Yang–Mills. The auxiliary term cannot supply a strictly positive pointwise mass floor on that stratum.*

Proof. The mixed plaquettes are the identity, so every term in [equation \(22\)](#) vanishes. The physical potential identity was proved in [equation \(34\)](#). Continuity places neighborhoods of the zero set in every positive-width low window. $\square$

5.3 Holonomy-twisted fiber spectrum

Fix a spatial link $e = (x, i)$ and define the unitary map on $\text{End}(\mathbb{C}^{d_\rho})$

$$\mathcal{H}_e(B) = \rho(H_x) B \rho(H_{x+i})^{-1}. \quad (48)$$

For a sequence $A = (A_0, \dots, A_{N-1})$, $N = N_{\tau, n}$, define

$$(D_{\tau, \mathcal{H}_e} A)_r = a_{\tau, n}^{-1} (A_{r+1} - A_r), \quad 0 \leq r < N-1, \quad (49)$$

$$(D_{\tau, \mathcal{H}_e} A)_{N-1} = a_{\tau, n}^{-1} (\mathcal{H}_e A_0 - A_{N-1}). \quad (50)$$

Theorem 5.5 (Exact holonomy-twisted spectrum). *Let $e^{i\theta_j}$, $-\pi < \theta_j \leq \pi$, be the eigenvalues of $\mathcal{H}_e$, counted with multiplicity. Then*

$$\text{Spec}(D_{\tau, \mathcal{H}_e}^* D_{\tau, \mathcal{H}_e}) = \left\{ \frac{4}{a_{\tau, n}^2} \sin^2 \left(\frac{2\pi k + \theta_j}{2N} \right) : 0 \leq k < N, j \right\}. \quad (51)$$

The kernel consists of constant sequences $A_r = A_0$ satisfying $\mathcal{H}_e A_0 = A_0$.

Proof. Choose an orthonormal eigenbasis b_j for the unitary $\mathcal{H}_e$. The vectors

$$A_{k,j}(r) = N^{-1/2} \exp \left(\frac{i(2\pi k + \theta_j)r}{N} \right) b_j$$

satisfy the twisted closing condition and diagonalize $D_{\tau, \mathcal{H}_e}$. The multiplier is

$$a_{\tau, n}^{-1} \left(\exp \left(\frac{i(2\pi k + \theta_j)}{N} \right) - 1 \right),$$

whose squared modulus is [equation \(51\)](#). It vanishes exactly for $k = 0$, $\theta_j = 0$, giving the stated kernel. $\square$

Corollary 5.6 (No compactification shortcut). *After removal of the exact kernel, no positive lower bound follows from compactness of the auxiliary cycle alone. Holonomy eigenphases $\theta_j \rightarrow 0$ produce positive eigenvalues in [equation \(51\)](#) tending to zero.*

Proof. Take $k = 0$ and nonzero $\theta_j \rightarrow 0$. The corresponding eigenvalue is asymptotic to $\theta_j^2 / (N^2 a_{\tau, n}^2)$. $\square$

5.4 Local zero spaces versus quantum ground spaces

A finite-block detectability argument normally begins with nonzero local ground-space projections. The continuous-group multiplication terms used here do not have such eigenspaces.

Proposition 5.7 (Classical local zeros have no nonzero L^2 ground vector). *Let p be any plaquette in a finite block containing at least one unconstrained link variable, and let*

$$V_p = M_{\ell_\rho(W_p)}$$

act on the corresponding product-Haar L^2 space or its gauge-invariant subspace. Then

$$\text{Ker } V_p = \{0\}, \quad \inf \text{Spec}(V_p) = 0. \quad (52)$$

The same conclusion holds for a finite nonempty sum of nonnegative plaquette multiplication terms.

Proof. Fix all plaquette links except one. As that link ranges over G , the plaquette holonomy ranges over G by a left-right translate. The condition $W_p = e$ therefore fixes that link to a single group element. Since G has positive dimension, a singleton has Haar measure zero. Fubini's theorem shows that the full zero set $\{W_p = e\}$ has product-Haar measure zero. By [theorem 2.1](#), a function in $\text{Ker } V_p$ must be supported on this null set, so it is zero in L^2 . On the other hand, every neighborhood of an identity configuration has positive measure and contains values of $\ell_\rho(W_p)$ arbitrarily close to zero; hence zero belongs to the essential range and is the spectral infimum. A sum can vanish only on the zero set of each term and in particular on the null set of one chosen term, so its kernel is also zero and its infimum is zero by neighborhoods of a flat configuration. $\square$

Remark 5.8 (Finite-block consequence). The classical flat configurations in [theorem 5.3](#) can be classified, but they do not furnish nonzero local quantum ground spaces for the multiplication operators. Finite-size and detectability methods [[19–22](#)] would first require an additional, rigorously controlled replacement such as spectral flattening, finite representation truncation, or a different local Hamiltonian decomposition. None of those replacements is part of the frozen regulator, and none may be inserted without a comparison theorem. In any event, no finite-block theorem can repair the algebraic failure in [theorem 1.1](#), which occurs on the exact global vacuum line.

5.5 Both low and high windows contain vacuum weight

Lemma 5.9 (Growth and nonconstancy of the mixed energy). *For every n , $\mathcal{K}_n$ is continuous, gauge invariant, and nonconstant, with minimum zero. Moreover*

$$\|K_n\| \geq 2\Lambda_* 2^{2n} \ell_\rho(g_*) \quad (53)$$

for any fixed $g_* \in G \setminus \{e\}$ and a suitable configuration. In particular $\|K_n\| \rightarrow \infty$.

Proof. The all-identity configuration has $\mathcal{K}_n = 0$. Set every auxiliary link to the identity and every spatial link to the identity except one link $U_i(x_0, \tau_0) = g_*$. The two mixed plaquettes adjacent to that link in the auxiliary direction have holonomies conjugate to g_* and g_*^{-1} ; all terms are nonnegative. Since $\ell_\rho(g_*^{-1}) = \ell_\rho(g_*)$,

$$\mathcal{K}_n \geq 2 \frac{a_n}{a_{\tau,n}^2 N_{\tau,n}} \ell_\rho(g_*) = 2\Lambda_* 2^{2n} \ell_\rho(g_*).$$

Continuity and gauge invariance were established in [theorem 3.1](#). $\square$

Theorem 5.10 (Vacuum overlap with every genuine multiplication split). *Fix $\varepsilon > 0$. For every sufficiently large n , let*

$$P = P_{\varepsilon,n}, \quad Q = I - P$$

on $\mathcal{N}_n$. Then

$$P\Omega_n \neq 0, \quad Q\Omega_n \neq 0. \quad (54)$$

More generally, equation (54) holds at any fixed n for which $0 < \varepsilon < \|K_n\|$.

Proof. The set $\{\mathcal{K}_n < \varepsilon\}$ contains an open neighborhood of the identity configuration and therefore has positive product-Haar measure. If $\varepsilon < \|K_n\|$, the set $\{\mathcal{K}_n > \varepsilon\}$ is also a nonempty open set and has positive measure. The continuous vacuum Ω_n is strictly positive everywhere by [theorem 3.3](#); hence its squared norm integrated over either set is strictly positive. Since $\mathcal{K}_n$ and Ω_n are auxiliary-translation invariant, both projected vectors lie in $\mathcal{N}_n$. The final assertion for all sufficiently large n follows from [equation \(53\)](#). $\square$

6 Vacuum-compatible Feshbach reduction

Fix n and work on $\mathcal{N}_n$. Let

$$P_{\Omega,n} = |\Omega_n\rangle\langle\Omega_n|. \quad (55)$$

For $\delta > 0$, put

$$r_{n,\delta} = e^{-a_{t,n}\delta} \quad (56)$$

and define the required vacuum-compatible operator

$$G_{n,\delta} = r_{n,\delta}I - T_n + (1 - r_{n,\delta})P_{\Omega,n}. \quad (57)$$

Proposition 6.1 (Exact transfer formulation of a gap). *The operator $G_{n,\delta}$ is bounded and self-adjoint,*

$$G_{n,\delta}\Omega_n = 0, \quad G_{n,\delta}|_{\Omega_n^\perp} = r_{n,\delta}I - T_n.$$

Consequently

$$G_{n,\delta} \geq 0 \iff H_n \geq \delta I \text{ on } \mathcal{N}_n \ominus \mathbb{C}\Omega_n. \quad (58)$$

Proof. The vacuum identity follows from $T_n\Omega_n = \Omega_n$. Self-adjointness of T_n makes $\mathbb{C}\Omega_n \oplus \Omega_n^\perp$ reducing, and the rank-one correction vanishes on the second summand. On a spectral value $t \in (0, 1]$ of T_n , the corresponding energy is $-a_{t,n}^{-1} \log t$. Since the exponential is decreasing, $t \leq e^{-a_{t,n}\delta}$ is equivalent to energy at least δ . $\square$

Fix $\varepsilon > 0$ and abbreviate

$$P = P_{\varepsilon,n}, \quad Q = I - P, \quad G = G_{n,\delta}, \quad r = r_{n,\delta}.$$

Whenever

$$D := QGQ \geq dQ \quad (d > 0), \quad (59)$$

define the vacuum-compatible Feshbach operator

$$F_{\varepsilon,n}^{\text{vac}}(\delta) = PGP - PGQD^{-1}QGP \text{ on } P\mathcal{N}_n. \quad (60)$$

Theorem 6.2 (Bounded Schur–Feshbach criterion). *Under equation (59),*

$$G \geq 0 \iff F_{\varepsilon,n}^{\text{vac}}(\delta) \geq 0. \quad (61)$$

Together with [theorem 6.1](#), either condition is equivalent to a gap at least δ for H_n on the neutral vacuum-orthogonal sector.

Proof. Relative to $\mathcal{N}_n = P\mathcal{N}_n \oplus Q\mathcal{N}_n$, write

$$G = \begin{pmatrix} A & C \\ C^* & D \end{pmatrix}.$$

For $x \in P\mathcal{N}_n$ and $y \in Q\mathcal{N}_n$, direct completion of the square gives

$$\begin{aligned} \langle x + y, G(x + y) \rangle &= \langle x, (A - CD^{-1}C^*)x \rangle \\ &\quad + \langle y + D^{-1}C^*x, D(y + D^{-1}C^*x) \rangle. \end{aligned} \quad (62)$$

The first operator is [equation \(60\)](#); the second term is nonnegative and can be made zero for each x . This proves the equivalence. $\square$

6.1 The transported vacuum line

Set

$$\omega = P\Omega_n, \quad \zeta = Q\Omega_n, \quad p = \|\omega\|^2, \quad q = \|\zeta\|^2 = 1 - p. \quad (63)$$

Proposition 6.3 (Vacuum transport through the high block). *Under [equation \(59\)](#), $\omega \neq 0$ and*

$$\zeta = -D^{-1}QG\omega. \quad (64)$$

Moreover

$$F_{\varepsilon,n}^{\text{vac}}(\delta)\omega = 0. \quad (65)$$

If

$$P_{\omega,n} = \frac{|\omega\rangle\langle\omega|}{\|\omega\|^2}, \quad P_{\varepsilon,n}^{\circ} = P - P_{\omega,n}, \quad (66)$$

then $P_{\varepsilon,n}^{\circ}$ reduces the self-adjoint Feshbach operator.

Proof. If $\omega = 0$, then $\Omega_n = \zeta \in Q\mathcal{N}_n$ and $D\Omega_n = QG\Omega_n = 0$, contradicting [equation \(59\)](#). The Q -component of $G(\omega + \zeta) = G\Omega_n = 0$ gives [equation \(64\)](#). The P -component gives

$$PGP\omega + PGQ\zeta = 0.$$

Substitution of [equation \(64\)](#) is exactly [equation \(65\)](#). Since the Feshbach operator is self-adjoint, a vector in its kernel has zero matrix element with its orthogonal complement; hence $\mathbb{C}\omega$ and $P_{\varepsilon,n}^{\circ}\mathcal{N}_n$ reduce it. $\square$

6.2 The exact obstruction

We now prove [theorem 1.1](#) and apply it to the frozen regulator.

Proof of theorem 1.1. Write $\alpha = 1 - r > 0$, $p = \|\omega\|^2 > 0$, and $\zeta = Q\Omega \neq 0$. The Q -component of $G\Omega = 0$ is

$$QG\omega + D\zeta = 0. \quad (67)$$

On the other hand,

$$QG\omega = -QT\omega + \alpha Q|\Omega\rangle\langle\Omega|\omega = -QT\omega + \alpha p\zeta.$$

Combining this identity with [equation \(67\)](#) gives

$$QT\omega = (D + \alpha pI)\zeta. \quad (68)$$

The Schur complement annihilates ω by the same calculation as in [theorem 6.3](#). Therefore

$$\langle \omega, PGP\omega \rangle = \langle QG\omega, D^{-1}QG\omega \rangle = \langle \zeta, D\zeta \rangle > 0. \quad (69)$$

Using [equation \(68\)](#),

$$\begin{aligned} \langle \omega, PTQD^{-1}QTP\omega \rangle &= \langle (D + \alpha pI)\zeta, D^{-1}(D + \alpha pI)\zeta \rangle \\ &= \langle \zeta, D\zeta \rangle + 2\alpha p\|\zeta\|^2 + \alpha^2 p^2 \langle \zeta, D^{-1}\zeta \rangle. \end{aligned} \quad (70)$$

Both additional terms are strictly positive because $\alpha > 0$, $p > 0$, $\zeta \neq 0$, and $D^{-1} > 0$. Comparing [equation \(70\)](#) with [equation \(69\)](#) proves [equation \(1\)](#). The quadratic-form inequality [equation \(2\)](#) evaluated on ω would give a contradiction for every $\eta \leq 1$. $\square$

Corollary 6.4 (Failure of requested item (R3) in the frozen family). *Fix $\varepsilon_0 > 0$, $\delta > 0$, and any sufficiently large n . Let $P = P_{\varepsilon_0, n}$ and $Q = I - P$. If the high block $D = QG_{n, \delta}Q$ is strictly positive, then no constant $\eta \leq 1$ satisfies*

$$PT_n QD^{-1}QT_n P \leq \eta PG_{n, \delta}P \quad (71)$$

on PN_n .

Proof. By [theorem 5.10](#), both $P\Omega_n$ and $Q\Omega_n$ are nonzero for all sufficiently large n . Apply [theorem 1.1](#). $\square$

Remark 6.5 (Exceptional choices are not the requested split). If $\varepsilon_0 \geq \|K_n\|$, then $Q = 0$ and there is no high block to invert; the proposed reduction collapses to the full global gap problem. If $\varepsilon_0 = 0$, the low multiplication projection is zero because the exact zero set has Haar measure zero by [theorem 5.7](#). The request specifies $\varepsilon_0 > 0$ independent of the regulator, and $\|K_n\| \rightarrow \infty$, so neither exceptional case avoids [theorem 6.4](#) along the frozen family.

Proposition 6.6 (A necessary vacuum-tail condition for direct high contraction). *Let $T = T^* \geq 0$, $T\Omega = \Omega$, and $P + Q = I$. If*

$$QTQ \leq \rho Q \quad (0 \leq \rho < 1), \quad (72)$$

then, with $p = \|P\Omega\|^2$,

$$p \geq \frac{1 - \rho}{2 - \rho}. \quad (73)$$

Proof. Let $\omega = P\Omega$ and $\zeta = Q\Omega$. From $T\Omega = \Omega$,

$$\langle \omega, T\zeta \rangle = p - \langle \omega, T\omega \rangle.$$

Expanding $1 = \langle \Omega, T\Omega \rangle$ and using self-adjointness gives

$$\langle \zeta, T\zeta \rangle = 1 - 2p + \langle \omega, T\omega \rangle \geq 1 - 2p,$$

because $T \geq 0$. The assumed high contraction gives

$$1 - 2p \leq \rho \|\zeta\|^2 = \rho(1 - p).$$

Rearrangement yields [equation \(73\)](#). $\square$

For $\rho = e^{-a_{t, n} m_Q}$, [equation \(73\)](#) requires the low window to contain at least a definite $O(a_{t, n})$ portion of the vacuum norm. This is a necessary condition only; it does not prove the high estimate.

6.3 The corrected punctured implication

The obstruction comes solely from applying the T -mixing inequality to the transported vacuum line. On the punctured low space the rank-one correction disappears.

Lemma 6.7 (Equality of the punctured T - and G -mixing forms). *Let $P^\circ = P_{\varepsilon,n}^\circ$. Then*

$$QGP^\circ = -QT_nP^\circ, \quad P^\circ GQ = -P^\circ T_nQ, \quad (74)$$

and

$$P^\circ GP^\circ = rP^\circ - P^\circ T_nP^\circ. \quad (75)$$

Proof. For $x \in \text{Ran } P^\circ$,

$$\langle \Omega_n, x \rangle = \langle P\Omega_n, x \rangle = \langle \omega, x \rangle = 0.$$

Thus $P_{\Omega,n}x = 0$. Since $Qx = 0$, the rI term has no off-diagonal block, proving [equation \(74\)](#). Compression to P° gives [equation \(75\)](#). $\square$

Proposition 6.8 (Corrected conditional two-window theorem). *Assume constants $m_0 > \delta > 0$, $m_Q > \delta$, and $0 \leq \eta < 1$ satisfy, at a fixed regulator,*

$$P^\circ T_n P^\circ \leq e^{-a_{t,n}m_0} P^\circ, \quad (76)$$

$$QT_nQ \leq e^{-a_{t,n}m_Q} Q, \quad (77)$$

$$P^\circ PT_nQ [QGQ]^{-1} QT_nPP^\circ \leq \eta P^\circ GP^\circ. \quad (78)$$

Then QGQ is invertible and

$$F_{\varepsilon,n}^{\text{vac}}(\delta) \geq (1 - \eta) \left(e^{-a_{t,n}\delta} - e^{-a_{t,n}m_0} \right) (P - P_{\omega,n}). \quad (79)$$

If the same constants hold for a family with $0 < a_{t,n} \leq a_t^{\max}$, then

$$F_{\varepsilon,n}^{\text{vac}}(\delta) \geq c a_{t,n} (P - P_{\omega,n}), \quad (80)$$

where the explicit regulator-independent constant is

$$c = (1 - \eta)(m_0 - \delta)e^{-a_t^{\max}m_0} > 0. \quad (81)$$

Proof. By [equation \(77\)](#) and positivity of the rank-one vacuum correction,

$$\begin{aligned} QGQ &= rQ - QT_nQ + (1 - r)|\zeta\rangle\langle\zeta| \\ &\geq \left(e^{-a_{t,n}\delta} - e^{-a_{t,n}m_Q} \right) Q > 0. \end{aligned}$$

Thus the inverse in [equation \(78\)](#) exists. By [theorem 6.7](#), the Feshbach compression is

$$\begin{aligned} P^\circ FP^\circ &= P^\circ GP^\circ - P^\circ PT_nQ (QGQ)^{-1} QT_nPP^\circ \\ &\geq (1 - \eta)P^\circ GP^\circ. \end{aligned}$$

The low contraction and [equation \(75\)](#) give

$$P^\circ GP^\circ \geq \left(e^{-a_{t,n}\delta} - e^{-a_{t,n}m_0} \right) P^\circ.$$

The Feshbach operator annihilates ω and has no cross term with P° by [theorem 6.3](#), proving [equation \(79\)](#). Finally,

$$\begin{aligned} e^{-a_{t,n}\delta} - e^{-a_{t,n}m_0} &= a_{t,n} \int_\delta^{m_0} e^{-a_{t,n}s} ds \\ &\geq a_{t,n}(m_0 - \delta)e^{-a_t^{\max}m_0}, \end{aligned}$$

which gives [equations \(80\) and \(81\)](#). $\square$

7 From the hard-window obstruction to a spectral-band regulator

The hard multiplication window of [section 6](#) is useful diagnostically: it identifies configurations with small mixed interruption energy and reveals the transported-vacuum obstruction. It is not a satisfactory local quantum decomposition. Two facts force a redesign.

First, [theorems 1.1](#) and [6.4](#) show that the unpunctured relative-mixing estimate is algebraically impossible whenever a nonconstant multiplication window splits the positive vacuum. Second, [theorem 5.7](#) shows that a nonnegative plaquette multiplication operator may have spectral infimum zero but no nonzero L^2 ground vector. Thus a frustration-free local-projector argument cannot begin from the exact zero set of the multiplication potential.

The replacement is to form a local Schrödinger operator in the auxiliary completion variables. Its kinetic term turns a classical well into a genuine quantum band. A separate holonomy-closure term removes continuous flat auxiliary moduli. The band projection, rather than the hard multiplication projection, becomes the object patched and eliminated in the low-energy proof.

7.1 Gauge-invariant holonomy closure

For a slice configuration, recall the auxiliary holonomy

$$H_x = V(x, 0)V(x, 1) \cdots V(x, N_{\tau, n} - 1).$$

Define

$$\mathcal{H}_n(V) = \sum_{x \in \Sigma_n} \ell_\rho(H_x). \quad (82)$$

Proposition 7.1 (Symmetry of the closure functional). *The function $\mathcal{H}_n$ is continuous, nonnegative, gauge invariant, and invariant under cyclic inner-time translation. It vanishes exactly when $H_x = e$ for every spatial site.*

Proof. Under a slice gauge transformation, $H_x \mapsto g(x, 0)H_x g(x, 0)^{-1}$, so conjugation invariance of ℓ_ρ gives gauge invariance. Changing the base point of the ordered product around the auxiliary cycle conjugates the holonomy by the transported link, so cyclic translation also preserves $\ell_\rho(H_x)$. Nonnegativity, continuity, and the characterization of the zero set follow from [theorem 2.1](#). $\square$

Use the frozen sequence $\kappa_{H, n}$ of [equation \(10\)](#). Define the exact completion-band potential

$$\mathcal{V}_n^{\text{cb}} = \mathcal{V}_n + \kappa_{H, n} \mathcal{H}_n. \quad (83)$$

On the gauge-invariant Sobolev form domain

$$\mathcal{Q}_n^{\text{sl}} = H^1(\mathcal{X}_n, \mu_n) \cap \mathcal{H}_n^{\text{sl}},$$

define

$$q_{n, 0}^{\text{cb}}[\Psi] = \|\mathcal{L}_n^{1/2} \Psi\|^2 + \langle \Psi, \mathcal{V}_n^{\text{cb}} \Psi \rangle. \quad (84)$$

Let $\mathbb{H}_{n, 0}^{\text{cb}}$ be the Friedrichs operator associated with this form, let

$$E_{0, n}^{\text{cb}} = \inf \text{Spec } \mathbb{H}_{n, 0}^{\text{cb}}, \quad H_n^{\text{cb}} = \mathbb{H}_{n, 0}^{\text{cb}} - E_{0, n}^{\text{cb}} I, \quad T_n^{\text{cb}} = e^{-a_{t, n} H_n^{\text{cb}}}. \quad (85)$$

Set $\mathcal{H}_n^{\text{cb}} := \mathcal{H}_n^{\text{sl}}$.

Proposition 7.2 (Exact finite-regulator completion Hamiltonian). *For every fixed regulator, $\mathbb{H}_{n,0}^{\text{cb}}$ is self-adjoint, bounded below, and has compact resolvent. Its lowest eigenvalue is simple and has a smooth strictly positive normalized eigenfunction Ω_n^{cb} . The shifted Hamiltonian H_n^{cb} is nonnegative and self-adjoint, and*

$$T_n^{\text{cb}} = e^{-a_{t,n} H_n^{\text{cb}}}$$

is self-adjoint, positive, injective, trace class, positivity improving, and satisfies $T_n^{\text{cb}} \Omega_n^{\text{cb}} = \Omega_n^{\text{cb}}$. Its ground-state transform gives a reversible two-sided vacuum Markov process and a finite-regulator OS reconstruction.

Proof. The first term in (84) is the closed elliptic Sobolev form $\|\mathcal{L}_n^{1/2} \Psi\|^2$, and the second is a bounded real multiplication-form perturbation on the compact configuration manifold. The representation theorem gives a self-adjoint semibounded operator with compact resolvent. The Trotter product formula, or equivalently the Feynman–Kac formula for a bounded potential on a compact manifold, shows that $e^{-t\mathbb{H}_{n,0}^{\text{cb}}}$ is positivity improving for every $t > 0$. Jentzsch–Perron theory therefore makes the ground eigenvalue simple with a smooth strictly positive eigenfunction. Subtracting $E_{0,n}^{\text{cb}}$ gives a nonnegative operator. Functional calculus gives injectivity and trace class of its positive-time semigroup. The ground-state transform and the proof of theorem 4.2 then produce the stationary reversible vacuum process and OS representation. $\square$

Define the completion-band vacuum probability measure and ground-state transform by

$$d\nu_n^{\text{cb}} = |\Omega_n^{\text{cb}}|^2 d\mu_n, \quad U_{\Omega,n}^{\text{cb}} f = \Omega_n^{\text{cb}} f, \quad \hat{T}_n^{\text{cb}} = (U_{\Omega,n}^{\text{cb}})^{-1} T_n^{\text{cb}} U_{\Omega,n}^{\text{cb}}. \quad (86)$$

Corollary 7.3 (Exact completion-band vacuum Markov representation). *The map*

$$U_{\Omega,n}^{\text{cb}} : L^2(\mathcal{X}_n, \nu_n^{\text{cb}})^{\mathcal{G}_n} \longrightarrow \mathcal{H}_n^{\text{cb}}$$

is unitary. The operator $\hat{T}_n^{\text{cb}}$ is a self-adjoint positivity-improving Markov contraction,

$$\hat{T}_n^{\text{cb}} \mathbf{1} = \mathbf{1},$$

and its reversible two-sided stationary process reconstructs the same vacuum Hilbert space and Hamiltonian H_n^{cb} .

Proof. Unitarity follows from $d\nu_n^{\text{cb}} = |\Omega_n^{\text{cb}}|^2 d\mu_n$. The eigenvalue equation $T_n^{\text{cb}} \Omega_n^{\text{cb}} = \Omega_n^{\text{cb}}$ gives the Markov identity, while unitary conjugacy preserves self-adjointness, positivity improvement, and the spectrum. The stationary-process and Osterwalder–Schrader conclusion is the ground-state-transform argument of theorem 4.2 with T_n, Ω_n replaced by $T_n^{\text{cb}}, \Omega_n^{\text{cb}}$. $\square$

Proposition 7.4 (Strang kernels approximate the exact semigroup at fixed regulator). *For fixed n and $t > 0$,*

$$\left(e^{-\frac{t}{2m} \mathcal{V}_n^{\text{cb}}} e^{-\frac{t}{m} \mathcal{L}_n} e^{-\frac{t}{2m} \mathcal{V}_n^{\text{cb}}} \right)^m \xrightarrow[m \rightarrow \infty]{s} e^{-t\mathbb{H}_{n,0}^{\text{cb}}}. \quad (87)$$

Every approximant has a strictly positive gauge-averaged heat kernel. No regulator-uniform rate in n is inferred from this fixed- n product formula.

Proof. The potential is bounded and the kinetic form is closed. The symmetric Trotter–Kato product formula gives strong convergence. Positivity of each factor and strict positivity of the heat kernel give the kernel statement. $\square$

Remark 7.5 (Universality and slow–fast burdens of the exact regulator). The exact Hamiltonian removes the earlier ambiguity between the logarithm of a one-step Strang operator and a separately introduced local Schrödinger operator. It does not by itself identify the local fiber Hamiltonians with components of the global form. A further slow–fast disintegration theorem is required to identify local completion fibers as components of the exact global form. The closure potential is an auxiliary regulator device, not part of four-dimensional Yang–Mills theory, so its induced local interactions must be proved irrelevant in a volume-uniform norm.

7.2 Design criteria for the new low-energy decomposition

The completion-band construction must satisfy five requirements.

- (B1) *Spectral reality*: local low spaces are ranges of spectral projections of self-adjoint completion Hamiltonians, not supports of measure-zero classical zero sets.
- (B2) *Gauge covariance*: band projections commute with local gauge constraints and with the exact compact regulator symmetries.
- (B3) *Transverse character*: the band removes only relative auxiliary modes; the physical τ -constant gauge field remains as a base variable.
- (B4) *Uniformity*: the transverse gap is bounded below in physical units uniformly in auxiliary resolution and physical refinement.
- (B5) *Universality*: eliminating the band produces pure four-dimensional Yang–Mills plus controlled irrelevant gauge-invariant corrections.

These criteria define the specifically DTT portion of the program.

8 Linear completion stiffness

The next lemma is the exact finite-dimensional core of the completion-band idea.

Lemma 8.1 (Cyclic Poincaré estimate). *Let E be a finite-dimensional real or complex Hilbert space, let $N \geq 2$, $a_\tau > 0$, and $\beta_\tau = Na_\tau$. For $\xi = (\xi_0, \dots, \xi_{N-1}) \in E^N$, set*

$$\bar{\xi} = \frac{1}{N} \sum_{r=0}^{N-1} \xi_r.$$

Then

$$\sum_{r=0}^{N-1} \frac{\|\xi_{r+1} - \xi_r\|^2}{a_\tau^2} \geq \frac{16}{\beta_\tau^2} \sum_{r=0}^{N-1} \|\xi_r - \bar{\xi}\|^2, \quad (88)$$

with cyclic indices.

Proof. Discrete Fourier transformation diagonalizes the left side. On the mode k , the eigenvalue is

$$4a_\tau^{-2} \sin^2\left(\frac{\pi k}{N}\right).$$

The constant mode is the mean. For $1 \leq k \leq N-1$, symmetry reduces to $1 \leq k \leq N/2$, and $\sin x \geq 2x/\pi$ on $[0, \pi/2]$ gives

$$4a_\tau^{-2} \sin^2(\pi/N) \geq 16(Na_\tau)^{-2}.$$

Summing the nonzero Fourier modes proves the result. $\square$

For the exact regulator, the quadratic expansion of the mixed potential around a coherent trivial-holonomy history has the normalized coefficient

$$\alpha = \alpha_n^{\text{mix}} = \frac{a_n}{2g_n^2 N_{\tau,n}}, \quad \kappa = \gamma_{H,n}^{\text{lin}} = \frac{1}{2} \kappa_{H,n} N_{\tau,n}. \quad (89)$$

Indeed, the mixed plaquette expansion contributes $\frac{a_n}{2g_n^2 N_{\tau,n}} \sum_r \|\xi_{r+1} - \xi_r\|^2 / a_{\tau,n}^2$, while

$$\kappa_{H,n} \ell_\rho \left(\exp \sum_r \xi_r \right) = \frac{1}{2} \kappa_{H,n} N_{\tau,n}^2 \|\bar{\xi}\|^2 + O(\|\xi\|^3).$$

Theorem 8.2 (Linearized transverse completion stiffness). *Let $\alpha, \kappa > 0$. Define on E^N*

$$q_{\alpha,\kappa}(\xi) = \alpha \sum_{r=0}^{N-1} \frac{\|\xi_{r+1} - \xi_r\|^2}{a_\tau^2} + \kappa N \|\bar{\xi}\|^2. \quad (90)$$

Then

$$q_{\alpha,\kappa}(\xi) \geq \min \left\{ \frac{16\alpha}{\beta_\tau^2}, \kappa \right\} \sum_{r=0}^{N-1} \|\xi_r\|^2. \quad (91)$$

Proof. The orthogonal decomposition $\xi_r = \bar{\xi} + (\xi_r - \bar{\xi})$ gives

$$\sum_r \|\xi_r\|^2 = N \|\bar{\xi}\|^2 + \sum_r \|\xi_r - \bar{\xi}\|^2.$$

The difference term annihilates the mean and satisfies [equation \(88\)](#) on the zero-mean part. The closure term is exactly κ times the squared norm of the mean component. Taking the smaller of the two coefficients proves [\(91\)](#). $\square$

Corollary 8.3 (Diverging frozen tangent band). *Under the frozen scaling in [\(10\)](#), the tangent normal-form floor is*

$$\Gamma_n^{\text{lin}} = \min \left\{ \frac{8\Lambda_*}{g_n^2}, \frac{\kappa_* \Lambda_*}{2} N_{\tau,n}^{\sigma_H} \right\} \longrightarrow \infty. \quad (92)$$

Every zero-mean relative mode is bounded below by $8\Lambda_*/g_n^2$, while the coherent mean/holonomy mode is bounded below by $(\kappa_* \Lambda_*/2) N_{\tau,n}^{\sigma_H}$.

Proof. Apply [theorem 8.2](#) with the coefficients in [equation \(89\)](#) and use [theorem 2.4](#). $\square$

This result is local and quadratic. It does not by itself control large excursions, gauge-slice transitions, multiple nonlinear wells, or coupling to the physical base variables.

9 Protected completion bands and their intrinsic connection

The local spectral projection used in an adiabatic reduction is itself geometric. This fact is standard, but it imposes a nontrivial auxiliary-invisibility condition: a degenerate completion band may carry a $U(m)$ connection even when no such extra field was intended [\[13–15\]](#).

Proposition 9.1 (Kato connection and curvature of an isolated band). *Let M be a smooth parameter manifold and let $u \mapsto C(u)$ be a C^1 self-adjoint family. Assume a fixed contour Γ encloses a finite spectral cluster of constant rank and define*

$$\Pi(u) = \frac{1}{2\pi i} \int_\Gamma (z - C(u))^{-1} dz.$$

Then $E = \text{Ran } \Pi \rightarrow M$ is a Hermitian vector bundle with metric connection

$$\nabla^\Pi s = \Pi ds. \quad (93)$$

In a local orthonormal frame e_a , its connection matrix is $\mathcal{A}_{ab} = \langle e_a, de_b \rangle$, its curvature is

$$\mathcal{F}^\Pi = \Pi(d\Pi \wedge d\Pi)\Pi, \quad (94)$$

and its holonomy is contained in $U(\text{rank } E)$.

Proof. The Riesz integral is a C^1 finite-rank projection of constant rank, hence defines a Hermitian subbundle. Projection of the exterior derivative gives a metric connection. Differentiating an orthonormal frame gives a skew-Hermitian local matrix, and the standard projection computation gives (94). $\square$

Remark 9.2 (Correct projection formula). The expression $\Pi d\Pi|_{\text{Ran } \Pi}$ is not the Kato connection: differentiating $\Pi^2 = \Pi$ gives $\Pi(d\Pi)\Pi = 0$. The intrinsic connection is Πd acting on sections. This distinction is important when derivative terms of the band projection enter an effective Hamiltonian.

Proposition 9.3 (Positive rank-one band and phase compatibility). *If the isolated band is rank one and has a globally defined normalized positive real eigenfunction $\phi(u)$, then its Abelian Berry one-form vanishes in that gauge. If a fixed orthogonal complex structure J commutes with every $C(u)$, then $[\Pi(u), J] = 0$, the Kato connection is J -linear, and every Feshbach reduction built from J -invariant forms is J -linear.*

Proof. For real normalized ϕ , differentiation of $\langle \phi, \phi \rangle = 1$ gives $\langle \phi, d\phi \rangle = 0$. Commutation of the resolvent with J gives commutation of the Riesz projection, and J -linearity follows from J being parameter independent. The Feshbach statement follows from the quadratic-form square identity. $\square$

Remark 9.4 (Retained carrier bands versus removable completion bands). A retained protected band can legitimately carry physical gauge holonomy. A completion band in the present paper is different: it is introduced only to eliminate auxiliary degrees of freedom. A successful pure-Yang–Mills limit must therefore prove that its Kato curvature vanishes, becomes scalar and topological, is absorbed into the declared gauge connection, or belongs to a sector whose energy diverges. Otherwise the regulator has generated an unintended finite-energy gauge field.

10 Adiabatic Schur–Feshbach elimination

This section records the operator and quadratic-form Schur–Feshbach identities used by the auxiliary-time model.

10.1 A block-operator theorem

Theorem 10.1 (Spectral Feshbach reduction below a transverse band). *Let $H = H^*$ on $\mathcal{H} = P\mathcal{H} \oplus Q\mathcal{H}$. Assume $P \text{Dom } H \subseteq \text{Dom } H$, write*

$$A = PHP, \quad B = PHQ, \quad D = QHQ,$$

and suppose B extends to a bounded operator with $\|B\| \leq b$. Suppose $D \geq \Gamma Q$ for some $\Gamma > 0$. For real $z < \Gamma$, define

$$F_H(z) = A - zP - B(D - zQ)^{-1}B^*. \quad (95)$$

Then:

(i) $H - z \geq 0$ if and only if $F_H(z) \geq 0$;

(ii)

$$0 \leq B(D - zQ)^{-1}B^* \leq \frac{b^2}{\Gamma - z}P; \quad (96)$$

(iii) whenever $F_H(z)$ is invertible, the block inverse is

$$(H - z)^{-1} = \begin{pmatrix} F^{-1} & -F^{-1}B(D - z)^{-1} \\ -(D - z)^{-1}B^*F^{-1} & (D - z)^{-1} + (D - z)^{-1}B^*F^{-1}B(D - z)^{-1} \end{pmatrix}, \quad (97)$$

where $F = F_H(z)$, and

$$\|Q(H - z)^{-1}\| \leq \frac{1}{\Gamma - z} + \frac{b}{\Gamma - z}\|F_H(z)^{-1}\| \quad (98)$$

$$+ \frac{b^2}{(\Gamma - z)^2}\|F_H(z)^{-1}\|. \quad (99)$$

Proof. For $x \in P\mathcal{H}$ and $y \in Q \operatorname{Dom} H$, completion of the square gives

$$\begin{aligned} \langle x + y, (H - z)(x + y) \rangle &= \langle x, F_H(z)x \rangle \\ &\quad + \langle y + (D - z)^{-1}B^*x, (D - z)[y + (D - z)^{-1}B^*x] \rangle. \end{aligned}$$

This proves (i). Functional calculus gives $(D - z)^{-1} \leq (\Gamma - z)^{-1}Q$, proving (ii). Solving the two block equations yields [equation \(97\)](#). The lower row contains a first block of norm at most $b(\Gamma - z)^{-1}\|F^{-1}\|$ and a second block of norm at most $(\Gamma - z)^{-1} + b^2(\Gamma - z)^{-2}\|F^{-1}\|$. The triangle inequality gives [equation \(98\)](#). $\square$

Corollary 10.2 (Low-block stability). *If $A \geq \Delta P$ and*

$$\frac{b^2}{\Gamma - z} < \Delta - z,$$

then $H \geq z$. In particular, if $\Gamma \rightarrow \infty$ while b remains bounded, the low spectrum converges to that of A on every fixed compact energy interval.

Proof. By [\(96\)](#),

$$F_H(z) \geq \left(\Delta - z - \frac{b^2}{\Gamma - z} \right) P > 0.$$

Apply [theorem 10.1](#). $\square$

Theorem 10.3 (Quadratic-form Feshbach identity). *Let q be a densely defined closed semibounded quadratic form on $\mathcal{H} = P\mathcal{H} \oplus Q\mathcal{H}$, and assume*

$$\mathcal{Q} := \operatorname{Dom} q = P\mathcal{Q} \oplus Q\mathcal{Q}.$$

For real z , set $q_z = q - z\|\cdot\|^2$ and write, using the polarized sesquilinear forms associated with the same symbols,

$$q_z[p + r] = a_z[p] + d_z[r] + 2 \operatorname{Re} b[p, r].$$

Assume d_z is closed and strictly positive on $Q\mathcal{Q}$, and that for each $p \in P\mathcal{Q}$ the functional $r \mapsto b[p, r]$ is continuous in the Hilbert norm $d_z[r]^{1/2}$. Let $K_z p \in Q\mathcal{Q}$ be the unique Riesz representative satisfying

$$d_z[K_z p, r] = b[p, r] \quad (r \in Q\mathcal{Q}), \quad (100)$$

and define the Feshbach form

$$f_z[p] = a_z[p] - d_z[K_z p]. \quad (101)$$

Then

$$q_z[p + r] = f_z[p] + d_z[r + K_z p]. \quad (102)$$

Consequently $q_z \geq 0$ if and only if $f_z \geq 0$. If $a_z[p] \geq 0$ and

$$d_z[K_z p] \leq \eta a_z[p] \quad (0 \leq \eta < 1), \quad (103)$$

then $f_z[p] \geq (1 - \eta)a_z[p]$.

Proof. By equation (100),

$$d_z[r + K_z p] = d_z[r] + 2 \operatorname{Re} b[p, r] + d_z[K_z p].$$

Subtracting the final term gives equation (102). For fixed p , the right side is minimized at $r = -K_z p$, proving the equivalence. The last assertion follows directly from equations (101) and (103). $\square$

Remark 10.4 (Why the form theorem is the operative one). For a gauge Schrödinger Hamiltonian the block PHQ need not be bounded. The natural small parameter is the norm of the cross form between the slow and fast form spaces. The bounded-operator theorem remains a useful benchmark, but theorem 10.3 is the correct interface for the exact regulator.

11 Scope, publication claim, and the first fast-sector problem

The results above are finite-regulator and operator theoretic. They establish the following implications without assuming a continuum target:

- mixed plaquette geometry $\longrightarrow$ a bounded interruption observable,
- positive vacuum plus a hard window $\longrightarrow$ an exact transported-vacuum obstruction,
- puncturing plus explicit contraction estimates $\longrightarrow$ a finite-regulator spectral margin,
- fast kinetic and closure scaling $\longrightarrow$ diverging linearized auxiliary stiffness.

They do not establish the contraction estimates uniformly in volume and lattice spacing, and they do not construct a nontrivial four-dimensional continuum theory.

A nonlinear completion-band theorem would require an orbit-type analysis of the local gauge quotient, a slow-fast disintegration of the exact global Friedrichs form, a uniform conditional Poincaré inequality in the fast variables, and control of the Kato curvature, Berry connection, and Born-Huang corrections described in section 9. The first concrete test is the following.

Open Problem 11.1 (Uniform $SU(2)$ fast sector). Fix $G = SU(2)$, one finite physical block, trivial magnetic and center sector, and the frozen scaling of equation (10). Construct the exact conditional auxiliary Hamiltonian from the global Friedrichs form and prove, uniformly in the physical base field u , that its ground state is unique and positive and that

$$E_{1,n}(u) - E_{0,n}(u) \geq \Gamma_{\tau,n}, \quad \Gamma_{\tau,n} \longrightarrow \infty.$$

Prove gauge-covariant derivative bounds on the ground projection strong enough that the induced Born-Huang form is $o(1)$ relative to the four-dimensional physical form. Prove that the completion-band Kato curvature satisfies one of the no-extra-gauge alternatives in theorem 9.4. Finally, show in a volume-independent local interaction norm that the closure-induced effective interactions tend to zero.

Success on [theorem 11.1](#) would show that the auxiliary sector supplies a genuine fast band while decoupling from low-energy physics. A Yang–Mills mass-gap proof would still require a separate four-dimensional nonabelian coercivity theorem, a nontrivial Osterwalder–Schrader continuum limit, and intrinsic Yang–Mills identification.

12 Conclusion

The hard-window obstruction is exact and model independent: a positive vacuum cut by a multiplication window cannot satisfy the natural unpunctured relative-mixing estimate. The punctured construction repairs the algebra, while the quadratic-form formulation makes the Schur–Feshbach method applicable to unbounded lattice gauge Hamiltonians. The protected-band section identifies the Kato connection that any nonlinear fast band will carry and converts absence of unintended gauge holonomy into a precise decoupling obligation. The auxiliary closure scaling exhibits a nonempty linearized fast-sector window. These results provide a technically clean starting point for a nonlinear completion-band analysis, without claiming that the four-dimensional Yang–Mills infrared problem has already been solved.

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